An Invitation to $\mathbb{T}^3$-Gowdy Spacetimes

Master's Thesis
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Abstract

This thesis is a small review of results on Gowdy spacetimes. While we assume familiarity with some fundamental concepts, our primary intention is to provide a friendly introduction for the student entering modern Mathematical General Relativity. The geometry and equation structure of $\mathbb{T}^3$-Gowdy symmetry make it an ideal pedagogical model and we hope that this thesis will serve as a helpful manual. The first chapter contains the origins of Gowdy spacetimes and its goal is to derive the metric under the Gowdy assumptions following [RG]. The second chapter deals with results related to big bang singularities and covers the work of Hans Ringström in [HR]. Finally, in the third chapter we make a partial exposition of the recent result regarding the BKL-proposal from [WL].

1. Introduction

The initial assumptions of the Gowdy symmetry are the following [RG]:

(a). Our spacetime $(\mathcal{M}, g)$ is a globally hyperbolic vacuum spacetime (that is, $Ric(g) = 0$) and is foliated by closed and orientable Cauchy hypersurfaces.

(b). The isometry group of the spacetime, $I(\mathcal{M})$, contains a compact, connected, two-dimensional Lie subgroup, say G.

(c). This subgroup G of $I(\mathcal{M})$ acts effectively (that is, if for all $p \in \mathcal{M}$, $g \cdot p = p$ then $g = e$) on the Cauchy hypersurfaces which are invariant under the action of the group.

(d). This subgroup of $I(\mathcal{M})$ generates two independent and spacelike Killing vector fields.

There is one more assumption but we postpone it since it is quite unnatural compared to the others listed above.

A theorem by Mostert asserts that such actions on 3-manifolds are only possible if our subgroup is either $\mathbb{T}^2$ or $SO(3)$. For dimensional reasons, $G = \mathbb{T}^2$. Moreover, the Cauchy hypersurfaces can only be one of the following:

1. $\mathbb{T}^3$,

2. $\mathbb{S}^3$ or

3. $\mathbb{S}^2 \times \mathbb{S}^1$

In this thesis we consider the case $\Sigma \simeq \mathbb{T}^3$.

2. Derivation

In this section we are interested in examining Robert Gowdy's original work on the derivation of a formula for the metric for spacetimes that satisfy the assumptions of the Gowdy symmetry.
The quadrad $(\mathcal{M}, \pi, \mathcal{M}/G, G)$ is a smooth Principal G-Bundle where $\mathcal{M}/G$ becomes a Lorentzian 2-manifold. This metric then can be pulled-back to $\mathcal{M}$ and decompose the metric on $\mathcal{M}$ as in :

$$g = \pi^*(\tilde{g}) + g|_{\mathcal{V}}$$

where $\mathcal{V}$ is the vertical distribution spanned by the action:

$$\mathcal{V} \coloneqq \bigcup_{p\in\mathcal{M}} V_p = \bigcup_{p\in\mathcal{M}} T_p(O_p),$$

where O_p is the orbit of $p \in \mathcal{M}$. The horizontal distribution is naturally defined in our case:

$$\mathcal{H} \coloneqq \mathcal{V}^{\perp}.$$

Since every 2-Lorentzian manifold is conformally flat we may find coordinates $(t, \theta) \in \mathcal{M}/G$ such that we have:

$$g = e^{2a}(-dt^2 + d\theta^2) + g|_{\mathcal{V}}.$$

The coordinates on $\mathcal{M}$ may look like $(t, \theta, \sigma, \delta) \in \mathcal{M}/G \times \mathbb{T}^2$ (locally) and a is a smooth function of (t, θ). The metric on $\mathcal{M}$, locally, now reads:

$$\begin{aligned} g = e^{2a}(-dt^2 + d\theta^2) &+ g_{22}(d\sigma + A^{\sigma}_t\, dt + A^{\sigma}_{\theta}\, d\theta)^2 \\ &+ g_{23}(d\sigma + A^{\sigma}_t\, dt + A^{\sigma}_{\theta}\, d\theta) \\ &\otimes (d\delta + A^{\delta}_t\, dt + A^{\delta}_{\theta}\, d\theta) \\ &+ g_{33}(d\delta + A^{\delta}_t dt + A^{\delta}_{\theta} d\theta)^2 \end{aligned}$$

We would like to get rid of the coefficients $A^{\sigma,\delta}_{t,\theta}$. The following theorem helps!

Theorem 2.1 *The following are equivalent:*

1. Locally the metric reads:

$$g = e^{2a}(-dt^2+d\theta^2)+g_{22}d\sigma^2+g_{23}(d\sigma\otimes d\delta+d\delta\otimes d\sigma)+g_{33}d\delta^2.$$

2. The horizontal distribution $\mathcal{H} \coloneqq \mathcal{V}^{\perp}$ is integrable.

3. The bundle is flat (curvature form vanishes).

4. The twist constants

$$\varepsilon_X \coloneqq *(X^{\flat} \wedge Y^{\flat} \wedge dX^{\flat})$$

and

$$\varepsilon_Y \coloneqq *(Y^{\flat} \wedge X^{\flat} \wedge dY^{\flat})$$

vanish.

The assumption that the twist constants vanish is the final assumption for the $\mathbb{T}^3$-Gowdy spacetime which is, in fact, an integrability condition.

3. Singularities

By considering the gradient of the determinant $R \coloneqq (\det(g_{ij})_{i,j=2,3})^{1/2}$ to be timelike, a change of coordinates and the derived structure of the metric allow us to consider $R = t = e^{-\tau}$. The metric, then, reads:

$$g = t^{-1/2}e^{\lambda/2}(-dt^2 + d\theta^2) + t(e^P(d\sigma + Qd\delta)^2 + e^{-P}d\delta^2),$$

or

$$\begin{aligned} g =& e^{(\tau-\lambda)/2}(-e^{2\tau}d\tau^2 + d\theta^2) \\ &+ e^{-\tau}\{e^P d\sigma^2 + e^P Q(d\sigma \otimes d\delta + d\delta \otimes d\sigma) \\ &+ (e^P Q^2 + e^{-P})d\delta^2\} \end{aligned}$$

In the second gauge, for $(\tau, \theta, \sigma, \delta) \in \mathbb{R} \times \mathbb{T}^3$, the Einstein equations read:

$$\partial^2_{\tau}P - e^{-2\tau}\partial^2_{\theta}P - e^{2P}((\partial_{\tau}Q)^2 - e^{-2\tau}(\partial_{\theta}Q)^2) = 0$$

$$\partial^2_{\tau}Q - e^{-2\tau}\partial^2_{\theta}Q + 2(\partial_{\tau}P\, \partial_{\tau}Q - e^{-2\tau}\partial_{\theta}P\, \partial_{\theta}Q) = 0$$

and

$$\partial_{\tau}\lambda = (\partial_{\tau}P)^2 + e^{-2\tau}(\partial_{\theta}P)^2 + e^{2P}((\partial_{\tau}Q)^2 + e^{-2\tau}(\partial_{\theta}Q)^2).$$

$$\partial_{\theta}\lambda = 2(\partial_{\theta}P\, \partial_{\tau}P + e^{2P}\partial_{\tau}Q\, \partial_{\theta}Q)$$

Hawking's incompleteness theorem indicates a singularity in the form of geodesic incompleteness. What is the nature of this singularity?

Numerical studies and heuristics indicate that the functions P and Q behave as follows :

$$\begin{cases} P(\tau, \theta) \sim v(\theta)\tau + \phi(\theta) + e^{-\varepsilon\tau}u(\tau, \theta) \\ \\ Q(\tau, \theta) \sim q(\theta) + e^{-2v(\theta)\tau}(\psi + w)(\tau, \theta) \end{cases}$$

for $\epsilon > 0$, $v(\theta) \in (0, 1)$ while $w, u \to 0$ as $\tau \to \infty$.

We cannot expect a curvature blow-up in the general situation. For example, the solution $P = \tau$, $Q = 0$, $\lambda = \tau$ admits a vanishing curvature tensor. But:
For sufficiently fast convergence we have the blow-up of the Kretschmann scalar as $\tau \to \infty$.
Consider the energy:

$$E_k(P, Q, \tau) \coloneqq \frac{1}{2}\int_{\mathbb{S}^1} e^{2P}(\partial_{\tau}\partial^k_{\theta}Q)^2 + e^{2P-2\tau}(\partial^{k+1}_{\theta}Q)^2\, d\theta.$$

Theorem 3.1 (Ringström, '02) *Let $(p_0, p_1, q_0, q_1) \in C^{\infty}(\mathbb{S}^1)$ initial data such that $2\gamma \leq p_1 \leq 1 - 2\gamma$ where $\gamma \in (0, 1)$. If τ_0 is big enough and $e_k(p_0, q_0, q_1, \tau_0)$ is a bound of $E_k(P, Q, \tau_0)$ for $k = 0, 1, 2$ that is also sufficiently small then there exist smooth solutions to the Gowdy system on $[\tau_0, \infty) \times \mathbb{S}^1$. Moreover, there exist smooth $v, w, q, r : \mathbb{S}^1 \to \mathbb{R}$ with $\gamma \leq v \leq 1 - \gamma$ and polynomials $\pi_{1,k}, \pi_{2,k}, \pi_{3,k}$ such that for all integers k we have the asymptotics:*

$$\begin{cases} \|P - v(\tau - \tau_0) - w\|_{C^k(\mathbb{S}^1)} \leq \pi_{1,k}e^{-2\gamma(\tau-\tau_0)} \\ \\ \|e^{2v(\tau-\tau_0)+w}(Q - q) + r/2v\|_{C^k(\mathbb{S}^1)} \leq \pi_{2,k}e^{-2\gamma(\tau-\tau_0)} \\ \\ \|\partial_{\tau}P - v\|_{C^k(\mathbb{S}^1)} + \|e^{2v(\tau-\tau_0)+w}\partial_{\tau}Q - r\|_{C^k(\mathbb{S}^1)} \leq \pi_{3,k}e^{-2\gamma(\tau-\tau_0)} \end{cases}$$

With the aforementioned asymptotics we have:

$$\lim_{\tau\to\infty} \inf_{\theta\in\mathbb{S}^1} |R_{abcd}R^{abcd}| = +\infty.$$

We conclude with a remarkable result:

Theorem 3.2 (Ringström, '05) *There exists a dense subset of the space of initial data for the Gowdy system, the solutions of which exhibit curvature blow-up in a dense subset of the singularity.*

4. Dynamics

By setting $t \coloneqq e^{-\tau} \in (0, \infty)$ we have the metric and the first two Einstein equations as in:

$$g = t^{-1/2}e^{\lambda/2}(-dt^2 + d\theta^2) + t(e^P(d\sigma + Qd\delta)^2 + e^{-P}d\delta^2),$$

$$\begin{cases} (t\partial_t)^2P - (t\partial_{\theta})^2P - e^{2P}((t\partial_tQ)^2 - (t\partial_{\theta}Q)^2) = 0, \\ \\ (t\partial_t)^2Q - (t\partial_{\theta})^2Q + 2t^2(\partial_tP\, \partial_tQ - \partial_{\theta}P\, \partial_{\theta}Q) = 0 \end{cases}.$$

The singularity now occurs as $t \to 0$.
Let $c : (0, t_0] \to (0, t_0] \times \mathbb{S}^1$ a timelike past-directed smooth curve, $c(t) \coloneqq (t, \theta(t))$. Moreover, set

$$\mathcal{P}_c(t) \coloneqq -t\partial_tP(c(t)),\ \mathcal{Q}_c(t) \coloneqq e^P t\partial_{\theta}Q(c(t))$$

Theorem 4.1 (W. Li, '24) *Consider $\gamma \in (0, 1)$ and a positive $\zeta > 0$. Suppose, then, initial data $(P_D, Q_D, \dot{P}_D, \dot{Q}_D) = (P, Q, t\partial_tP, t\partial_tQ)|_{t=t_0}$ that satisfy:*

$$-\dot{P}_D > \gamma,\ (1 + \dot{P}_D)^2 + (e^{P_D}t_0\partial_{\theta}Q_D)^2 < (1 - \gamma)^2$$

and some L^2-norms/energies of sufficiently large order (which depends on γ) are bounded by ζ. If t_0 is chosen sufficiently small (depending on γ and ζ) then the functions $\mathcal{P}_c$ and $\mathcal{Q}_c$ satisfy:

$$t\frac{d}{dt}\mathcal{P}_c = \mathcal{Q}^2_c + \textbf{\textit{error}},\quad t\frac{d}{dt}\mathcal{Q}_c = (1 - \mathcal{P}_c)\mathcal{Q}_c + \textbf{\textit{error}}$$

where both error terms vanish as $t \to 0$. Moreover, we have $\mathcal{P}_c(t) \to \mathcal{P}_{c,\infty}$ and $\mathcal{Q}_c(t) \to 0$ as $t \to 0$. If either the limit $\lim_{t\to 0} \partial_{\theta}Q(c(t))$ does not exist or exists but is nonzero then $\mathcal{P}_{c,\infty} \leq 1$ and:

$$|\mathcal{P}_{c,\infty} - \min\{\mathcal{P}_c(t_0), 2 - \mathcal{P}_c(t_0)\}| \lesssim (t_0^{\gamma'} + \mathcal{Q}_c(t_0))^{1/2}.$$

The last inequality may be interpreted as a **BKL-bounce** which is explained by the following study:

In 1971, BKL proposed an ansatz for the metric as it approaches the singularity for $M \coloneqq (0, T) \times \Sigma$:

$$g = -d\tau^2 + \sum_{i=1}^{3} \tau^{2p_i(x)}\omega_i(x) \otimes \omega_i(x) + \dots$$

where $p_i,\ i = 1, 2, 3$ have to satisfy the Kasner relations:

$$p_1 + p_2 + p_3 = p_1^2 + p_2^2 + p_3^2 = 1.$$

Moreover, they assumed that $\partial_{\tau} >> \partial_x$. This ansatz is unstable as one approaches the singularity ($\tau \to 0$).
This instability is viewed through the bounces of the exponents when $p_2 < 0$ and $\omega_2 \wedge d\omega_2 \neq 0$:

$$p_1 \mapsto \frac{p_1 + 2p_2}{1 + 2p_2},\ p_2 \mapsto \frac{-p_2}{1 + 2p_2},\ p_3 \mapsto \frac{p_3 + 2p_2}{1 + 2p_2}.$$

In the case of Gowdy symmetry, after an appropriate change of coordinates (in order for the metric to look like the BKL ansatz), the bounce is modeled by the transition:

$$V(\theta) \mapsto 2 - V(\theta),$$

when $V(\theta) > 1$ and $q'(\theta) \neq 0$, where V is the pointwise limit of $-t\partial_tP$ as $t \to 0$ while q is the limit of Q.

References

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