Numerical approximation of the derivative nonlinear Schrödinger equation

Nikolaos Vougiantzis

Graduate Program: *"Mathematics and its Applications*

Department of Mathematics and Applied Mathematics

ΠΑΝΕΠΙΣΤΗΜΙΟ ΚΡΗΤΗΣ
UNIVERSITY OF CRETE

Abstract

The solution to an initial- and Dirichlet boundary-value problem of the *Derivative Nonlinear Schrödinger* (DNLS) equation is approximated by Crank-Nicolson finite difference (FDM) method that conserves the discrete L^2 norm. We provide an optimal second order error estimate in the discrete L^2 norm, assuming k, h and $\frac{k^2}{h}$ are sufficiently small, where k is the time step and h is the space mesh size.

The DNLS equation

For $T > 0$, $\Omega = [\gamma, \delta]$ and $Q = [0, T] \times \Omega$. We seek $\phi : Q \to \mathbb{C}$ solving the following initial - and Dirichlet boundary - value problem for the derivative non-linear Schrödinger (DNLS) equation:

$$\begin{aligned} &\phi_t = i\,\alpha\,\phi_{xx} + \rho\left(|\phi|^2\phi\right)_x + f, \quad (0,T] \times [\gamma, \delta], \\ &\phi(t,\gamma) = \phi(t,\delta) = 0, \quad t \in [0,T], \\ &\phi(0,x) = \phi_0(x), \quad x \in [\gamma, \delta]. \end{aligned} \tag{1}$$

Here, i is the imaginary unit, α and ρ are non-zero real constants, $\phi_0 : Q \to \mathbb{C}$ with $\phi_0(\gamma) = \phi_0(\delta) = 0$ and $f : Q \to \mathbb{C}$.

The Finite Difference Method (FDM)

For $n = 0, ..., N$ approximate the vector $\phi^n \in \mathbb{C}_0^{J+2}$ with vector $\Phi^n \in \mathbb{C}_0^{J+2}$ defined by

Step 1: Set

$$\Phi^0 := \phi^0.$$

Step 2: For $n = 0, ..., N-1$, find $\Phi^{n+1} \in \mathbb{C}_0^{J+2}$ such that

$$\partial_k \Phi^n = i\,a\,\Delta_h \Phi^{n+\frac{1}{2}} + \frac{\rho}{2}\mathcal{L}(\Phi^{n+\frac{1}{2}}) + f^{n+\frac{1}{2}}, \tag{2}$$

where $\mathcal{L} : \mathbb{C}_0^{J+2} \to \mathbb{C}_0^{J+2}$ is the non-linear discrete operator defined by

$$\mathcal{L}(v) := \partial_h(|v|^2 \otimes v) + \partial_h(|v|^2) \otimes v + |v|^2 \otimes \partial_h v,$$

and $f^{n+\frac{1}{2}} \in \mathbb{C}_0^{J+2}$ with $f_j^{n+\frac{1}{2}} := f(t_{n+\frac{1}{2}}, x_j)$ for $j = 1, ..., J$.

Proposition 1 (Existence) *For any given $\Phi^0 \in \mathbb{C}_0^{J+2}$ there exist finite difference approximations $(\Phi^n)_{n=1}^N \subseteq \mathbb{C}_0^{J+2}$.*

Proposition 2 (Uniqueness) *Let $M > 0$ be a constant.Then, there exists a constant $C_{U2} > 0$, independent of k and h, such that: if $C_{U2} M^2 k < 1$ and there are (FDM) approximations $(\Phi^m)_{m=0}^N \subseteq \mathbb{C}_0^{J+2}$ satisfying $\max_{0 \le m \le N} \|\Phi^m\|_{1,\infty,h} \le M$, then they are unique.*

Consistency

For $m = 0, ..., N-1$, the *consistency error* $R^m \in \mathbb{C}_0^{J+2}$ of the (FDM) method at the time level $t = t_{m+\frac{1}{2}}$ is defined by

$$\partial_k \phi^m = i a \Delta_h \phi^{m+\frac{1}{2}} + \tfrac{\rho}{2}\mathcal{L}\left(\phi^{m+\frac{1}{2}}\right) + f^{m+\frac{1}{2}} + R^m.$$

Proposition 3 (Consistency error) *Let ϕ be the solution to the problem (1).If $\phi \in C_{t,x}^{3,4}(Q)$ and $\partial_x \phi, \partial_x^2 \phi \in C_{t,x}^{2,0}(Q)$, then it holds that*

$$\max_{0 \le m \le N-1} \|R^m\|_\infty \le C_{cs}(h^2 + k^2),$$

where C_{cs} is a positive constant independent of k and h, and depends on ϕ and the derivatives of ϕ.

Convergence

For $\lambda > 0$, let $\xi_\lambda : [0, +\infty] \to [0, 1]$ be a continuous function defined by

$$\xi_\lambda(x) := \begin{cases} 1 \text{ if } x \le \lambda \\ \frac{2\lambda - x}{\lambda} \text{ if } x \in (\lambda, 2\lambda], \\ 0 \text{ if } x > 2\lambda \end{cases} \quad \forall x \in \mathbb{R},$$

Then, for $\lambda > 0$ and $t \in [0, T]$, we define an operator $\mathfrak{m}_\lambda(t; \cdot) : \mathbb{C}_0^{J+2} \to \mathbb{C}_0^{J+2}$ by

$$\begin{aligned} \mathfrak{m}_\lambda(t; w) = w\,\xi_\lambda\left(\|w - \Lambda_h(\phi(t,\cdot))\|_{1,\infty,h}\right) \\ + \Lambda_h(\phi(t,\cdot))\left[1 - \xi_\lambda\left(\|w - \Lambda_h(\phi(t,\cdot))\|_{1,\infty,h}\right)\right], \\ \forall w \in \mathbb{C}_0^{J+2}, \end{aligned}$$

where ϕ is the solution to the problem (1).
For $\lambda > 0$, we introduce a modified finite difference scheme following the steps below:

Step 1: Set

$$S_\lambda^0 := \phi^0.$$

Step 2: For $n = 0, ..., N-1$, find $S_\lambda^{n+1} \in \mathbb{C}_0^{J+2}$ such that

$$\partial_k S_\lambda^n = i\,a\,\Delta_h S_\lambda^{n+\frac{1}{2}} + \frac{\rho}{2}\mathcal{L}_\lambda(t_{n+\frac{1}{2}}; S_\lambda^{n+\frac{1}{2}}) + f^{n+\frac{1}{2}}.$$

where

$$\begin{aligned} \mathcal{L}_\lambda(t; v) = \partial_h\left(|\mathfrak{m}_\lambda(t; v)|^2 \otimes v\right) + \partial_h\left(|\mathfrak{m}_\lambda(t; v)|^2\right) \otimes v \\ + |\mathfrak{m}_\lambda(t; v)|^2 \otimes \partial_h v. \end{aligned}$$

Proposition 4 *Let $\lambda \ge \lambda_0$.Then, there exists constant $C_{MEX} > 0$, independent of k, h and λ, such that:if $k\lambda^2 C_{MEX} \le 1$, then the for any $S_\lambda^0 \in \mathbb{C}_0^{J+2}$ there exist modified finite difference approximations $(S_\lambda^m)_{m=1}^N \subseteq \mathbb{C}_0^{J+2}$.*

Theorem 1 (Error estimation) *Let $\phi \in C_{t,x}^{3,4}(Q)$, with $\partial_x \phi, \partial_x^2 \phi \in C_{t,x}^{2,0}(Q)$, be the solution to the problem (1). Also let $\lambda_* = 1 + \lambda_0$. Then, there exists a positive constant C_{MVC1}, such that if $C_{MCV1}\lambda_* k^2 \le 1$ and $(S_{\lambda_*}^n)_{n=0}^N$ be modified finite difference approximations, then*

$$\max_{0 \le n \le N} \|\phi^n - S_{\lambda_*}^n\|_h \le C_{MCV2}(k^2 + h^2)$$

$$\max_{0 \le n \le N} |\phi^n - S_{\lambda_*}^n|_{1,h} \le C_{MCV3}(k + h)$$

where C_{MCV2} and C_{MCV3} positive constants independent of k and h.

In the following Theorem we establish that, under a mild condition, the modified approximations are (FDM) approximation and hence they share the same properties.

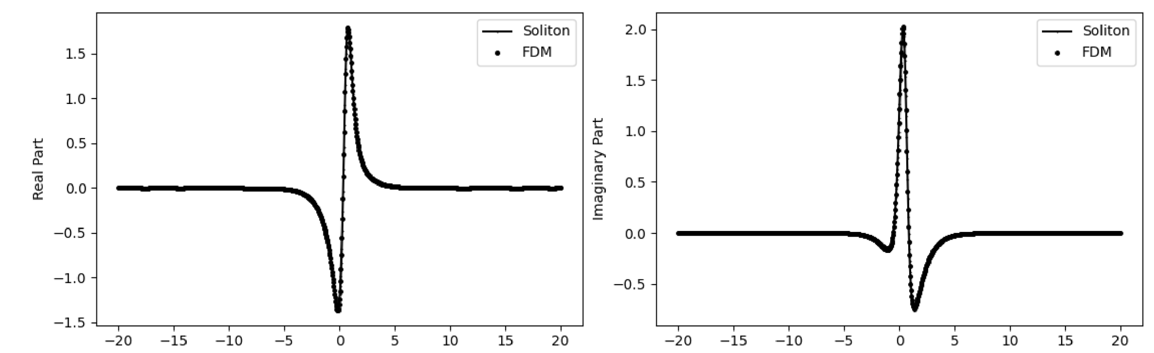
Theorem 2 *Let us assume that $\phi \in C_{t,x}^{3,4}(Q)$ and $\phi_x, \phi_{xx} \in C_{t,x}^{2,0}(Q)$. Also, let $\lambda_* = 1 + \lambda_0$, $\left(S_{\lambda_*}^n\right)_{n=0}^N$ be modified finite difference approximations and $C_{MCV1}\lambda_*^2 k \le 1$. Then, there exists positive constants $C_{CG1} \ge C_{MCV1}$ and C_{GC2}, independent of k and h, such that: if $C_{CG1}\lambda_*^2 k \le 1$ and $C_{GC2}(k^2 + kh^{-\frac{1}{2}} + h^{\frac{3}{2}}) \le \lambda_*$, then the modified finite difference approximations $\left(S_{\lambda_*}^n\right)_{n=0}^N$ are unique, bounded in the discrete $W^{1,\infty}$ norm and they are (FDM) approximations, i.e. for $\Phi^n = S_{\lambda_*}^n, \quad n = 0, ..., N$ (2) holds.*

Numerical Results

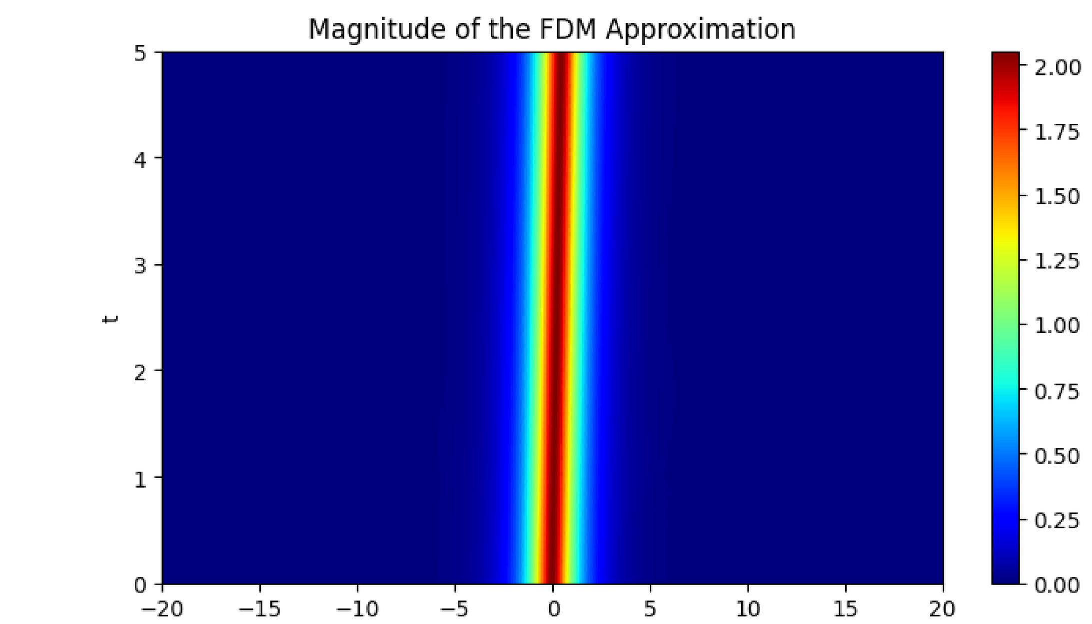
We consider the problem (1) with $T = 5$, $[\gamma, \delta] = [-20, 20]$, $\alpha = 1$, $\rho = -1$, $f = 0$ and the single soliton solution (see, e.g., **[8]**)

$$u(x,t) = -\frac{\left(-1+\left(\frac{1}{160}-\frac{i}{8}\right)e^{t/5-2x+2}\right)}{\left(-1+\left(\frac{1}{160}+\frac{i}{8}\right)e^{t/5-2x+2}\right)^2} e^{1+\left(\frac{1}{10}+\frac{399i}{400}\right)t+\left(-1+\frac{i}{20}\right)x}$$

ν	$E_\infty(\nu,\nu)$	**Rate**	$E_1(\nu,\nu)$	**Rate**	$\max_{0\le n\le N} \|\Phi^n\|_{1,\infty,h}$
1000	1.73496e-01	–	6.4106e-01	–	3.1747
2000	4.35346e-02	1.9947	1.6093e-01	1.9940	3.1807
4000	1.08835e-02	2.0000	4.0239e-02	1.9997	3.1819
8000	2.72295e-03	1.9989	1.0066e-02	1.9991	3.1822
16000	6.79854e-04	2.0019	2.5141e-03	2.0013	3.1823
32000	1.69554e-04	2.0035	6.2715e-04	2.0031	3.1823



Approximating the soliton solution at $T = 5$ with $N = J = 1000$.



Magnitude of the finite difference approximation with $N = J = 1000$.

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