Integrability of Hamiltonian Systems by Thimm's Method

Master's Thesis
Georgios Vlachakis
Supervisor: Konstantin Athanassopoulos
Department of Mathematics and Applied Mathematics, University of Crete, Heraklion, Greece

Abstract

This work is devoted to Thimm's method for proving integrability of Hamiltonian systems, such as geodesic flows of Riemannian manifold. Thimm's work allows the construction of families of first integrals in involution for Hamiltonian systems which are invariant under the Hamiltonian action of a Lie group G. This is applied to invariant Hamiltonian systems on the tangent bundles of certain homogeneous spaces. In particular, we show the integrability of the geodesic flows of real Grassmannian manifold.

1. Hamiltonian systems

Let (M,ω) be a symplectic $2n$-manifold. A smooth vector field X on M is called $Hamiltonian$ if there exists a smooth function $H : M \to \mathbb{R}$ such that $i_X\omega = dH$. In other words,

$$\omega_p(X_p, v_p) = v_p(H)$$

for every $v_p \in T_pM$ and $p \in M$. We usally write $X = X_H$. Since $L_X\omega = d(i_X\omega) = 0$, the flow of X_H consists of symplectomorphisms.

Definition. If (M,ω) is a symplectic manifold and $F,\ G \in C^\infty(M)$, then the smooth function

$$\{F,G\} = i_{X_G} i_{X_F}\omega \in C^\infty(M)$$

is called the $Poisson\ bracket$ of F and G.

Proposition. *A smooth function $F : M \to \mathbb{R}$ on a symplectic manifold M is a first integral of a Hamiltonian vector field X_H on M if and only if $\{F,H\} = 0$.*

2. Hamiltonian Lie group action

Definition. Let (M,ω) be a symplectic manifold and G a Lie group. A smooth group action $\phi : G \times M \to M$ is called $symplectic$ if $\phi_g = \phi(g,\cdot) : M \to M$ is a symplectomorphism for every $g \in G$. The symplectic action is called *Hamiltonian* if each fundamental vector field $\phi_*(X)$, $X \in \mathfrak{g}$, is Hamiltonian. The action is called $Poisson$ if there is a Lie algebra homomorphism $\rho : \mathfrak{g} \to C^\infty(M)$ such that $\phi_*(X) = X_{\rho(X)}$.

Definition. Let (M,ω) be a connected, symplectic manifold, G a Lie group with Lie algebra $\mathfrak{g}$. A *momentum map* for a Poisson action ϕ is a smooth map $\mu : M \to \mathfrak{g}^*$ such that $\rho : \mathfrak{g} \to C^\infty(M)$ defined by $\rho(X)(p) = \mu(p)(X)$ for $X \in \mathfrak{g}$ and $p \in M$ satisfies

(i) $\phi_*(X) = X_{\rho(X)}$,
(ii)$\{\rho(X), \rho(Y)\} = \rho([X,Y])$ for every $X, Y \in \mathfrak{g}$.

3. Completely integrable Hamiltonian systems

Let (M,ω) be a connected, symplectic $2n$-manifold and $H_1 \in C^\infty(M)$. The Hamiltonian vector field X_{H_1} is called *completely integrable* if there are $H_2,...,H_n \in C^\infty(M)$ such that $\{H_i, H_j\} = 0$ for every $1 \le i,j \le n$ and the differential 1-forms $dH_1, dH_2, ..., dH_n$ are linearly independent on a dense open set $D \subset M$.

4. Symplectic geometry on $T(G/K)$

Let K be a closed subgroup of a connected Lie group G and the set $\{\sigma K \mid \sigma \in K\}$ of left cosets be the homogeneous space $M := G/K$. Let also $\pi : G \to G/K$ denote the natural projection $\pi(\sigma) = \sigma K$. We consider homogeneous spaces that posses the following property.

Property A. *On the Lie algebra $\mathfrak{g}$ of G there exists an Ad_G-invariant, symmetric, non-degenerate billinear form B such that the restriction of B to the Lie algebra $\mathfrak{k}$ of K is non-degenerate.*

This property leads to the existence of $\mathfrak{m}$, the B-complement of $\mathfrak{k}$ in $\mathfrak{g}$. $B_{\mathfrak{k}\times\mathfrak{k}}$ defines a G-invariant metric on M. We Identify $\mathfrak{g}^*$ with $\mathfrak{g}$ using B and T^*M with TM by means of the invariant metric defined by B.

We derive a formula for the momentum map P on TM of a homogeneous space assuming it satisfies property (A).

Lemma. *Let M=G/K be a homogeneous space satisfying property (A). Then the momentum map $P : TM \to \mathfrak{g}$ is given by $P(g\xi) = \mathrm{Ad}_g\xi$, where $g \in G$, $\xi \in \mathfrak{m}$, $g\xi = (L_g)_{*\pi(e)} \circ \pi_{*e}(\xi) \in T_{\pi(g)}M$.*

5. The symplectic structure of the tangent bundle of homogeneous spaces

Proposition. *Let M=G/K be a homogeneous space satisfying property* (A). *Identifying $T_\xi TM$, $\xi \in \mathfrak{m}$, by means of the exponential map with the subspace*

$$\{(v_1 - \frac{1}{2}[v_1,\xi] + w)_{(e,\xi)} \mid v, w \in \mathfrak{m}\}$$

of $G \times \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g}$, then the symplectic structure of $TM \in g\xi$ is given by

$$\omega_{g\xi}(g_{*\xi}(v_1, -\tfrac{1}{2}[v_1,\xi]+w_1), g_{*\xi}(v_2, -\tfrac{1}{2}[v_2,\xi]+w_2)) = B(v_1,w_2) - B(v_2,w_1).$$

6. Hamiltonian systems on the tangent bundle of homogeneous spaces

Proposition. *Let $h : \mathfrak{m} \to \mathbb{R}$ be Ad_K-invariant and $f : TM \to \mathbb{R}$ the G-invariant Hamiltonian defined by h. The Hamiltonian vector field X_f of f is given by the formula*

$$X_f(g\xi) = g_{*\xi}(v_1, -\frac{1}{2}[v_1,\xi] + w_1)$$

where $v_1 = \mathrm{grad}h(\xi)$ and $w_1 = -\frac{1}{2}[\mathrm{grad}h(\xi), \xi]_{\mathfrak{m}}$.
If $f_1, f_2 : TM \to \mathbb{R}$ are two invariant Hamiltonians defined by $h_1, h_2 : \mathfrak{m} \to \mathbb{R}$ respectively, then their Poisson bracket is

$$\{f_1, f_2\}(g\xi) = -B([\mathrm{grad}h_1(\xi), \mathrm{grad}h_2(\xi)], \xi).$$

G-invariant Hamiltonian systems on TM have many first integrals such as all functions $f = h \circ P$ for some smooth function $h : \mathfrak{g} \to \mathbb{R}$ and $P : TM \to \mathfrak{g}$ the momentum map. We compute the Hamiltonian vector field of $f = h \circ P$.

Proposition. *Let M=G/K be a homogeneous space satisfying property (A) and $h : \mathfrak{g} \to \mathbb{R}$. Then the Hamiltonian vector field X_f of $f = h \circ P$ is given by the formula*

$$X_f(g\xi) = g_{*\xi}(v, -\frac{1}{2}[v,\xi] + w)$$

where $v = \mathrm{Ad}_{g^{-1}}(\zeta)_{\mathfrak{m}}$, $w = [\mathrm{Ad}_{g^{-1}}(\zeta), \xi]_{\mathfrak{m}} - \frac{1}{2}[\mathrm{Ad}_{g^{-1}}(\zeta)_{\mathfrak{m}}, \xi]_{\mathfrak{m}}$ and $\zeta = \mathrm{grad}h(\mathrm{Ad}_g(\xi))$.

7. First integrals in involution from non-degenerate Lie subalgebras

The Poisson bracket in $\mathfrak{g}^*$ transforms to the Poisson bracket in $\mathfrak{g}$ defined by

$$\{h_1, h_2\}(\xi) = B(\xi, [\mathrm{grad}h_1(\xi), \mathrm{grad}h_2(\xi)])$$

for $\xi \in \mathfrak{g}$ and $h_1, h_2 \in C^\infty(\mathfrak{g})$, where the gradients are considered with respect to B. We find that h is Ad_G-invariant if and only if $[\xi, \mathrm{grad}h(\xi)] = 0$ for all $\xi \in \mathfrak{g}$.

Suppose that $\mathfrak{g}', \mathfrak{g}'' \subset \mathfrak{g}$ are subalgebras of $\mathfrak{g}$ so that $B|_{\mathfrak{g}'\times\mathfrak{g}'}$ and $B|_{\mathfrak{g}''\times\mathfrak{g}''}$ are non-degenerate. There are B-orthogonal projections $\pi' : \mathfrak{g} \to \mathfrak{g}'$ and $\pi'' : \mathfrak{g} \to \mathfrak{g}''$. If $h' : \mathfrak{g}' \to \mathbb{R}$ is an $\mathrm{Ad}_{G'}$-invariant smooth function and $h'' : \mathfrak{g}'' \to \mathbb{R}$ is an $\mathrm{Ad}_{G''}$-invariant smooth function, the Poisson bracket of $h' \circ \pi'$ and $h'' \circ \pi''$ is

$$\{h' \circ \pi', h'' \circ \pi''\}(\xi) = B(\xi_{\mathfrak{g}'^{\perp}}, [\mathrm{grad}h'(\pi'(\xi)), \mathrm{grad}h''(\pi''(\xi))]).$$

So if $[\mathfrak{g}', \mathfrak{g}''] \subset \mathfrak{g}'$, then $\{h' \circ \pi', h'' \circ \pi''\} = 0$. This holds in particular if $\mathfrak{g}'' \subset \mathfrak{g}'$.

Lemma. *Let (M,ω) be a symplectic manifold with a Poisson action of the Lie group G on M. Let $\mu : M \to \mathfrak{g}^*$ be the corresponding momentum map. If $h_1, h_2 : \mathfrak{g}^* \to \mathbb{R}$ are smooth functions, then*

$$\{h_1 \circ \mu, h_2 \circ \mu\} = \{h_1, h_2\} \circ \mu.$$

Proposition. *Let (M,ω) be a symplectic manifold with a Poisson action of the Lie group G on M with momentum map $\mu : M \to \mathfrak{g}^*$. Suppose that there exists an Ad_G-invariant, non-degenerate, symmetric bilinear form $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ on the Lie algebra $\mathfrak{g}$ of G. If*

$$\mathfrak{g}_1 \subset \mathfrak{g}_2 \subset ... \subset \mathfrak{g}_k \subset \mathfrak{g}_{k+1} = \mathfrak{g}$$

is a chain of non-degenerate (with respect to B) Lie subalgebras of $\mathfrak{g}$ and $h_i \in C^\infty(\mathfrak{g}_i)$, $1 \le i \le k+1$ and Ad-invariant functions, then $h_i \circ \pi_i \circ \mu$, $1 \le i \le k+1$ are first integrals in involution of X_F for every G-invariant Hamiltonian $F : M \to \mathbb{R}$, where $\pi_i : \mathfrak{g} \to \mathfrak{g}_i$ is the B-orthogonal projection, $1 \le i \le k+1$ and with respect to the identification of $\mathfrak{g}$ and $\mathfrak{g}^*$ defined by B.

8. Integrabilty of the geodesic flow of the real Grassmann manifolds

The real Grassmann manifold of p-planes in $\mathbb{R}^{n+1}$ is the homogeneous symmetric space

$$G_{p,q}(\mathbb{R}) = SO(n+1,\mathbb{R})/S(O(p,\mathbb{R}) \times O(q,\mathbb{R}))$$

$$= O(n+1,\mathbb{R})/(O(p,\mathbb{R}) \times O(q,\mathbb{R}))$$

where $p+q = n+1$, normalized by $p \le q$. The Killing form on the Lie algebra $\mathfrak{g} = \mathfrak{so}(n+1,\mathbb{R})$ leads to the Ad_G-invariant non-degenerate, symmetric, bilinear form on $\mathfrak{g}$

$$B(\xi,\eta) = -\frac{1}{2}\mathrm{Tr}(\xi \cdot \eta) = \frac{1}{2}\mathrm{Tr}(\xi\eta^t)$$

Let $K = S(O(p) \times O(p))$ with corresponding Lie algebra $\mathfrak{k}$. The complement of $\mathfrak{k}$ in $\mathfrak{g}$ is

$$\mathfrak{m} = \left\{ \begin{pmatrix} 0 & X \\ -X^t & 0 \end{pmatrix} \mid X \in \mathbb{R}^{p\times q} \right\}.$$

On $\mathfrak{g}$ we consider the polynomial functions $f_k : \mathfrak{g} \to \mathbb{R}$, $k = 1,2,...,p$ defined by

$$f_k(\xi) = -\frac{1}{4k}\mathrm{Tr}(\xi^{2k})$$

which are Ad_G-invariant. In particular their restriction on $\mathfrak{m}$ are Ad_K-invariant.

Proposition. *The polynomial functions $h_k : \mathfrak{so}(p+q,\mathbb{R}) \to \mathbb{R}$ with*

$$h_k(\xi) = \mathrm{Tr}\xi^{2k},\ k = 1,2,...,p,$$

where $p,q \in \mathbb{N}$, $p \le q$, are $SO(p+q,\mathbb{R})$-invariant with gradients

$$\mathrm{grad}h_k(\xi) = -2k\xi^{2k-1},$$

with respect to the metric $\langle X,Y \rangle = \mathrm{Tr}(XY^t)$. Moreover at any ξ in a maximal abelian subspace $\mathfrak{a} \subset \mathfrak{m}$, their gradients are tangent to $\mathfrak{a}$ and are linearly independent if ξ is a regular element of $\mathfrak{so}(p+q,\mathbb{R})$.

Now we consider the following chain

$$\mathbb{R}^1 \times \mathbb{R}^1 \subset \mathbb{R}^1 \times \mathbb{R}^2 \subset \mathbb{R}^2 \times \mathbb{R}^2 \subset ... \subset \mathbb{R}^p \times \mathbb{R}^{q-1} \subset \mathbb{R}^p \times \mathbb{R}^q$$

of subspaces of $\mathbb{R}^{n+1} = \mathbb{R}^p \times \mathbb{R}^q$. From this chain of vector subspaces we obtain the chain of Lie subgroups

$$O(1+1,\mathbb{R}) \subset O(1+2,\mathbb{R}) \subset ... \subset O(p+q-1,\mathbb{R}) \subset O(n+1,\mathbb{R}).$$

We can obtain a finite sequence of regular vectors $\xi_i \in \mathfrak{m}_i$ such that $\pi_i(\xi_j) = \xi_i$ for $i < j$ where $\pi_i : \mathfrak{so}(n+1,\mathbb{R}) \to \mathfrak{g}_i$ is the orthogonal projection.

If $P : TG_{p,q}(\mathbb{R}) \to \mathfrak{so}(n+1,\mathbb{R})$ denotes the representation of the momentum map induced by the metric, we have a total number of

$$2\frac{(p-1)p}{2} + p(q-p+1) = pq$$

first inegrals $F_{ij} = f_j \circ \pi_i \circ P : TG_{p,q}(\mathbb{R}) \to \mathbb{R}$ which are in involution. Their gradients are linearly independent at $\xi_n = \xi_{p+q-1}$ and so are their corresponding Hamiltonian vector fields. Since P is real analytic and $f_j \circ \pi_i$ are polynomial functions, all the F_{ji} are real analytic functions and so their gradients are linearly independent and so are their corresponding Hamiltonian vector fields.

Theorem. *The geodesic flow of the real Grassmanian $G_{p,q}(\mathbb{R})$ is completely integrable with pq real analytic functions on $TG_{p,q}(\mathbb{R})$ as a complete family of first integrals in involution.*

References

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