Infinite sequence of Ramanujan graphs and the Kadison Singer problem

Georgios Mavrogiannis

Supervisor: Mihalis Kolountzakis

Department of Mathematics and Applied Mathematics, University of Crete

gm615@cam.ac.uk

Abstract

We start our presentation by giving the necessary definitions and theorems that are in the core of the last two results. We define regular graphs, bipartite graphs and ramanujan graphs before we give the conjecture of Bilu-Linial. Afterwards we define the common interlacing, the interlacing family before we give the statement of the Kadison-Singer problem.

Introduction

Bilu and Linial suggested building an infinite suquence of d-regular bipartite Ramanujan graphs. Their work was based on the work of Eliot Lieb [**??**]. With similar theorems and technics the Kadison Singer problem was solved. This problem arose in the work of Paul Dirac in quantum mechanics in the 1940s. However after that it arose in various areas of mathematics, physics as well as computer science.

d-Regular graphs

A graph is an ordered pair $G = (V, E)$ comprising a set V of vertices and a set E of edges. In a common sense these vertices have a relation with one another which is represented by the edges, i.e. if two vertices are related they are connected with an edge, and we say that these vertices are neighbors. A **d-regular** is a graph that each vertex has d edges leaving it.

Bipartite graphs

Let's suppose an ordered pair $G = (V, E)$. We say that our graph G is bipartite if the vertices can be divided into two disjoint sets A and B, such that each vertex in A have no neighbors in A and each vertex in B has no neighbors in B.

Adjacency matrix

We need to formalize this intuitive construction that we named a graph. One way to formalize it is with the use of the adjacency matrix. Let's suppose that we have n vertices. The adjacency matrix is a transpose matrix which in the (i, j) entry has the number of edges joining the vertices i and j. It contains all the information we need about a graph.

Characteristic polynomial of a graph

The characteristic polynomial of a graph is just the characteristic polynomial of its adjacency matrix.

Ramanujan graph

Given an ordered pair $G = (V, E)$ where G is a d-regular graph, we can easily construct its adjacency matrix and find its characteristic polynomial. When the non trivial eigenvalues of the characteristic polynomial of a graph are in the interval $[-2\sqrt{d-1}, 2\sqrt{d-1}]$ we say that this graph is $Ramanujan$. The trivial eigenvalues are $-d, +d$

Conjecture of Bilu-Linial

For every $d \geq 3$ there is an infinite sequence of d-regular bipartite Ramanujan graphs

Common interlacing

Let g,f be polynomials of the form $g(x) = \prod_{i=1}^{n-1}(x - \alpha_i)$ and $f(x) = \prod_{i=1}^{n}(x - \beta_i)$. If

$$\beta_1 \leq \alpha_1 \leq \beta_2 \leq \alpha_2 \leq \dots \alpha_{n-1} \leq \beta_n$$

then we say that g **interlaces** f.
If $f_1, \dots f_k$ are of the same form then, we say that they have a **common interlacing** if $\exists$ a polynomial g which interlaces f_i, for all i.

Interlacing Family

Let $S_1, \dots, S_m$ be finite sets and $\forall s_1, \dots, s_m \in S_1 \times \dots \times S_m$ let $f_{s_1,\dots,s_m}$ be a real rooted polynomial of degree n, with positive leading coefficient. $\forall k < m$ define:

$$f_{s_1,\dots,s_k} = \sum_{s_{k+1}\in S_{k+1},\dots,s_m\in S_m} f_{s_1,\dots,s_k,s_{k+1},\dots,s_m}$$

as well as

$$f_{\emptyset} = \sum_{s_1\in S_1,\dots,s_m\in S_m} f_{s_1,\dots,s_m}$$

We say that the polynomials $\{f_{s_1,\dots,s_m}\}_{s_1,\dots,s_m}$ form an interlacing family if $\forall k = 0, \dots, m-1$ and all $s_1, \dots, s_k \in S_1 \times \dots \times S_k$, the polynomials

$$\{f_{s_1,\dots,s_k,t}\}_{t\in S_{k+1}}$$

have a common interlacing

Basic theorem about interlacing families

Let $S_1, \dots, S_m$ be finite sets and let $\{f_{s_1,\dots,s_m}\}_{s_i\in S_i}$ be an interlacing family of polynomials. Then, there exist some $(s_1, \dots, s_m) \in S_1 \times \dots \times S_m$ so that: $maxroot(f_{s_1,\dots,s_m}) \leq maxroot(f_{\emptyset})$

Kadison Singer problem

For every $r \in \mathbb{N}$ and $u_1, \dots u_m \in \mathbb{C}$ such that:
$\sum_{i=1}^{m} u_i u_i^\star = I$ and $|u_i|^2 \leq \delta$ Then:
$\exists$ a partitition $S_1, \dots S_r$ of $[m]$ such that

$$|| \sum_{i\in S_j} u_i u_i^\star || \leq (\frac{1}{\sqrt{r}}) + \sqrt{\delta})^2$$

References

[1] Eliot Lieb,Ole Heilmann *Theory of monomer dimer systems*, Springer, Developments in Mathematics (Book 19), (2009).

[2] Y.Bilu,N.Linial *Lifts,discrepancy and nearly optimal spectral gap.*,Combinatorica,26(5): 2006.

[3] *P.G.Casazza, D.Edidin, D.Kalra, V.I.Paulsen Projections and the Kadison-Singer Problem. Operators and matrices*

[4] *C.A.Akemann and J.Anderson Lyapunov theorems for operator alebras. Number 458 in Memoirs of the American Mathematical Society. AMS, 1991*

[5] *Extensions, restrictions, and representations of states on $C^\star$-algebras. Transactions of the American Mathematical Society, 249(2):303-329, 1979*

[6] *Extreme points in sets of positive linear maps on $\mathbb{B}(\mathbb{H})$. Journal of Functional Analysis*

[7] *A conjecture concerning the pure states of $\mathbb{B}(\mathbb{H})$ and related theorems.Topics in modern operator theory, Birkhaeuser, Basel-Boston, Mass, pages 27-43, 1981.*

[8] *Restricted invertibility of matrices and applications. In Analysis at Urbana, Volume 2, Analysis in Abstract Spaces, number 138 in London Mathematical Society Lecture Note Series, pages 61-107. Cambridge University Press, 1989*

[9] *Frames and the Feichtinger conjecture. Proceedings of the Mathematical Society, 133(4):1025-1033, 2005*

[10] *The Kadison-Singer problem in mathematics and engineering. Proceedings of the National Academy of Sciences of the United States of America, 103(7):2032-2039, 2006*

[11] *The Kadison-Singer problem in discrepancy theory. Discrete Mathematics, 278(1-3):227-239, 2004.*