WIGNER REPRESENTATION OF THE BERRY-BALAZS WAVE PACKET

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Introduction

In this poster we summarize the results of a graduate thesis dealing with the phase-space representation of the semi-classical evolution of the BB wave packet through its Wigner transform. The BB wave packet is the solution of the free Schrödinger equation with initial data the Airy function, and it has infinite L^2 norm. A class of Airy-type wave packets with finite L^2 norm is constructed by convolving the BB wave packet with certain square integrable functions. It is shown that in the classical limit, the Wigner transform of any wave packet in this class concentrates on the "Lagrangian manifold" of the BB wave packet.

The Berry-Balazs (BB) wave packet

The BB wave packet is the solution of the free Schrödinger equation (zero potential $V \equiv 0$)

$$i\varepsilon\partial_t\Psi^\varepsilon(x,t) = -\frac{\varepsilon^2}{2}\partial_x^2\Psi^\varepsilon(x,t)\ , \quad x\in\mathbb{R}\ ,\ t>0,$$

with initial data

$$\Psi_0^\varepsilon(x) = \Psi_{BB0}^\varepsilon(x) := Ai\Big(\frac{Bx}{\varepsilon^{2/3}}\Big)\ ,\ x\in\mathbb{R}\ ,$$

where $Ai(x)$ is the Airy function, and $B>0$ is a scaling parameter. The Airy function is defined by the integral

$$Ai(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i\left(z^3/3+zx\right)}dz \quad \notin L^2(\mathbb{R})$$

Solving the above IVP by Fourier transform, we derive

$$\Psi_{BB}^\varepsilon(x,t) = Ai\Big(\frac{B}{\varepsilon^{2/3}}\Big(x-\frac{B^3t^2}{4}\Big)\Big)e^{i\frac{B^3t}{2\varepsilon}\left(x-\frac{B^3t^2}{6}\right)}\ .$$

Geometrical optics of the BB wave packet

The standard asymptotic expansion of the Airy function, implies that the initial data $\Psi_{BB0}^\varepsilon(x)$ admit a *two-phase WKB expansion*, for $x<0$, $|x|$ large, and small ε, and they are exponentially small for $x>0$. The appropriate application of the geometrical optics (WKB) method, provides the *rays*

$$\overline{x}(t;q) = S_0'(q)t+q\ ,\quad S_0(q) = -\frac{2}{3}(-Bq)^{3/2}\ ,\quad q<0\ .$$

which are relevant to the IVP for the Schrödinger equation. The Jacobian of these rays $x=\overline{x}_-(t;q)$ vanishes along the *caustic* (red line in the figure below)

$$\mathcal{C}: x = x_c(t) := \overline{x}(t; q=-B^3t^2/4) = \frac{B^3t^2}{4}\ ,\quad t>0\ .$$

By eliminating the initial position q from the equations of the rays and their momenta $\overline{k}(t;q) = d\overline{x}(t;q)/dt$, we derive the moving "Lagrangian manifold (curve)"

$$\Lambda_t = \Big\{(x,k)|k^2+B^3(x-kt)=0\Big\}\ ,$$

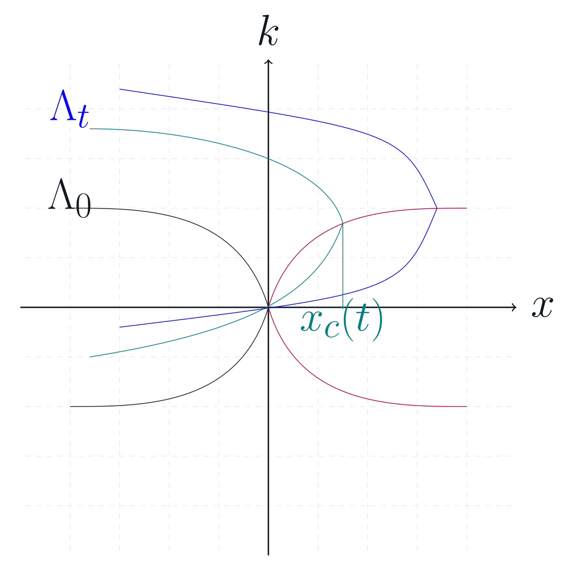
which is the image of the "initial Lagrangian manifold (curve)"

$$\Lambda_0 = \Big\{(q,p)|p^2+B^3q=0\Big\}\ ,$$

under the Hamiltonian flow $(x=pt+q\ ,\quad k=p)$ in the phase space. This flow is defined by the *bicharacteristics* $(x(t;q,p),k(t;q,p))$ of the *Hamiltonian system:*

$$\begin{cases} \frac{dx(t;q,p)}{dt} = k(t;q,p)\ , & x(0;q,p)=q, \\ \frac{dk(t;q,p)}{dt} = -V'\big(x(t;q,p)\big)\equiv 0\ , & k(0;q,p)=p \end{cases}$$

The rays are the projections of the bicharacteritics with initial momentum $p=S_0'(q)$ on to the physical space-time $\mathbb{R}^2_{xt}$.



The Wigner transform and the Wigner equation

We define the *Wigner transform* $W[f](x,k)$ of a complex-valued function $f(x)$ as

$$W[f](x,k) := \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ik\sigma}f\big(x+\frac{\sigma}{2}\big)\overline{f}\big(x-\frac{\sigma}{2}\big)d\sigma$$

and the corresponding *scaled Wigner transform* as:

$$W^\varepsilon[f](x,k) := \frac{1}{\varepsilon}W[f]\Big(x,\frac{k}{\varepsilon}\Big) = \frac{1}{\pi\varepsilon}\int_{-\infty}^{\infty}e^{-i\frac{2k}{\varepsilon}\sigma}f(x+\sigma)\overline{f}(x-\sigma)d\sigma$$

For the Schrödinger equation

$$i\varepsilon\partial_t\Psi^\varepsilon(x,t) = -\frac{\varepsilon^2}{2}\partial_x^2\Psi^\varepsilon(x,t)+V(x)\Psi^\varepsilon(x,t)\ ,\quad x\in\mathbb{R}\ ,\ t>0,$$

when the potential $V(x)$ is smooth, it turns out that $W^\varepsilon(x,k,t) = W^\varepsilon[\Psi^\varepsilon](x,k,t)$ satisfies the *Wigner equation*

$$\partial_tW^\varepsilon+k\partial_xW^\varepsilon-V'(x)\partial_kW^\varepsilon = \sum_{m=1}^{\infty}\alpha_m\varepsilon^{2m}V^{(2m+1)}(x)\partial_k^{(2m+1)}W^\varepsilon$$

where $\alpha_m = (-1)^m2^{-2m}/(2m+1)!\ ,\quad m=1,2,\dots$

In the simple case $V\equiv 0$, that we consider here, given the initial data $W^\varepsilon(q,p,0) = W_0^\varepsilon(q,p)$, the Wigner equation is a simple transport equation, and its solution is given by

$$W^\varepsilon(x,k,t) = W_0^\varepsilon(x-kt,k)$$

Wigner representation of the BB wave packet

The solution of the Wigner equation, with $V=0$ and initial data

$$W_{BB0}^\varepsilon(x,k) = W^\varepsilon[\Psi_{BB0}^\varepsilon](x,k) = \frac{2^{2/3}B}{\varepsilon^{2/3}}Ai\Big(\frac{2^{2/3}B}{\varepsilon^{2/3}}\big(\frac{k^2}{B^3}+x\big)\Big)\ ,$$

is given by the equation

$$W_{BB}^\varepsilon(x,k,t) = \frac{2^{2/3}B}{\varepsilon^{2/3}}Ai\Big(\frac{2^{2/3}B}{\varepsilon^{2/3}}\big(\frac{k^2}{B^3}+x-kt\big)\Big)\ ,$$

which is, in fact, the scaled Wigner transform $W^\varepsilon[\Psi_{BB}^\varepsilon](x,k,t)$ of the BB wave packet.

By the distribuitional limit $\frac{1}{\sigma}h\big(\frac{x}{\sigma}\big)\to\delta(x)\int_{\mathbb{R}}h\,dx$ as $\sigma\to 0$, it can be shown that

$$W_{BB0}^\varepsilon(x,k)\rightharpoonup W_{BB0}^0(x,k) = \delta\Big(k^2+B^3x\Big)\ ,\quad\text{as}\quad\varepsilon\to 0\ ,$$

and

$$W_{BB}^\varepsilon(x,k,t)\rightharpoonup W_{BB}^0(x,k,t) := \delta\Big(\frac{k^2}{B^3}+x-kt\Big)\ ,\quad\text{as}\quad\varepsilon\to 0\ .$$

We observe that the support of $W_{BB0}^0(x,k)$ (resp. $W_{BB}^0(x,k,t)$) is Λ_0 (resp. Λ_t), and that $W_{BB}^0(x,k,t)$ is the shift of $W_{BB0}^0(x,k)$ by the Hamiltonian flow..

Airy-type wave packets with finite energy

A class of Airy-type wave packets with finite L^2 norm is constructed by using L^2 initial data of the form

$$\Psi_0^\varepsilon(x) = \Psi_{0g}^\varepsilon(x) := g(x;\sigma)*\Psi_{BB0}^\varepsilon(x)\ ,$$

where g is any function of the form

$$g(x;\sigma) = \frac{1}{\sigma}f\big(\frac{x}{\sigma}\big)\ ,\quad f\in L^1(\mathbb{R})\cap L^2(\mathbb{R})\ ,\ \sigma>0\ .$$

The Wigner transform of Ψ_{0g}^ε is

$$W^\varepsilon[\Psi_{0g}^\varepsilon](x,k) = \frac{2^{2/3}B}{\varepsilon^{2/3}}\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{i\left(\frac{z^3}{3}+\frac{2^{2/3}B}{\varepsilon^{2/3}}\left(\frac{k^2}{B^3}+x\right)z\right)} \times\widehat{g}\Big(\frac{k}{\varepsilon}+\frac{Bz}{2^{1/3}\varepsilon^{2/3}}\Big)\overline{\widehat{g}}\Big(\frac{k}{\varepsilon}-\frac{Bz}{2^{1/3}\varepsilon^{2/3}}\Big)dz\ .$$

where $\widehat{g}$ is the Fourier transform of g. Assuming that $\widehat{g}$ is smooth, we derive the series expansion

$$W^\varepsilon[\Psi_{0g}^\varepsilon](x,k) = 2\mu\sum_{\ell=0}^{\infty}C_\ell^\varepsilon(\widehat{g})\mu^\ell\ \frac{1}{2\pi}\int_{-\infty}^{+\infty}e^{i\left(z^3/3+2\mu\theta(x,k)z\right)}z^\ell dz\ ,$$

where

$$\mu = \frac{B}{2^{1/3}\varepsilon^{2/3}}\ \text{and}\quad\theta(x,k) = \frac{k^2}{B^3}+x\ ,$$

and

$$C_0^\varepsilon(\widehat{g}) = \Big|\widehat{g}\Big(\frac{k}{\varepsilon}\Big)\Big|^2\ ,\quad C_1^\varepsilon(\widehat{g}) = -2i\Im\left(\overline{\widehat{g}}\Big(\frac{k}{\varepsilon}\Big)\widehat{g}\,'\Big(\frac{k}{\varepsilon}\Big)\right)\ ,\dots$$

The integrals can be expressed in terms of the Airy function, and the series is written in the form

$$W^\varepsilon[\Psi_{0g}^\varepsilon](x,k) = \sum_{\ell=0}^{\infty}\frac{C_\ell^\varepsilon(\widehat{g})}{2^\ell}\frac{\partial^\ell}{\partial\theta^\ell}\Big(2\mu Ai\big(2\mu\theta(x,k)\big)\Big)$$

Now we observe that as $\varepsilon\to 0$, with $\sigma=\sigma(\varepsilon)=\varepsilon^{1+\gamma}$, for any $\gamma>0$, the coefficients C_ℓ^ε have well defined limits

$$C_0^\varepsilon(\widehat{g}) = \Big|\widehat{g}\Big(\frac{k}{\varepsilon}\Big)\Big|^2\to\Big|\widehat{f}(0)\Big|^2\ ,\quad C_\ell^\varepsilon(\widehat{g})\to 0\ ,\quad\ell=2,3,\dots$$

Moreover, since the distributional relation

$$\frac{1}{\sigma}h\big(\frac{x}{\sigma}\big)\to\delta(x)\int_{\mathbb{R}}h\,dx\ \text{as}\quad\sigma\to 0\ ,$$

can be differentiated any number of times, we derive that

$$\frac{\partial^\ell}{\partial\theta^\ell}\Big(2\mu Ai\big(2\mu\theta(x,k)\big)\Big)\to\delta^{(\ell)}(\theta(x,k))\ ,\ \text{as}\quad\varepsilon\to 0\ .$$

Therefore, the *formal weak limit* of $W^\varepsilon[\Psi_{0g}^\varepsilon]$ is

$$W^\varepsilon[\Psi_{0g}^\varepsilon](x,k)\rightharpoonup\Big|\widehat{f}(0)\Big|^2\delta\Big(\frac{k^2}{B^3}+x\Big)\ ,\quad\text{as}\quad\varepsilon\to 0\ .$$

We conclude that for the considered class of finite-energy wave packets, the limit Wigner function is always concentrated on the "Lagrangian manifold" Λ_0 of the BB wave packet.

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