The LlogL inequality for the strong maximal function

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Abstract

Throughout this project we work on $\mathbb{R}^d$ with the Lebesgue measure and by the term rectangle we will always mean rectangle with sides parallel to the coordinate axes. We investigate the interaction between the covering properties of a basis, the differentiation properties of a basis and the size of the associated maximal operator. After giving some general results, we focus on the basis $\mathcal{R}$ consisting of all the rectangles on $\mathbb{R}^d$ and the strong maximal operator M_s. This basis differentiates $L^p(\mathbb{R}^d)$, $p > 1$, but it does not differentiate $L^1(\mathbb{R}^d)$. The determining difference between this basis and the one consisting of all balls (or cubes), is that the volume of a ball is comparable with its diameter, whereas the volume of a rectangle can be arbitrary small while its diameter is arbitrary large. This is the reason why differentiation of $L^1(\mathbb{R}^d)$ fails for $\mathcal{R}$. It is also known that instead of $L^1(\mathbb{R}^d)$, $\mathcal{R}$ differentiates the function space $L(1+\log^+ L)^{d-1} = \{f : \mathbb{R}^d \to \mathbb{R}^d \ \textit{measurable} : \int |f|(1+\log^+|f|)^{d-1} < +\infty\}$, which in some way is optimal. The purpose of this thesis is to give a geometric proof of this assertion, imitating the proof of Lebesgue differentiation theorem. More specifically, we shall present a suitable covering lemma for rectangles (as a substitute of Besicovitch covering lemma which fails), which with standard techniques leads to the so called *LlogL* inequality for the strong maximal function, by which differentiation of $L(1+\log^+ L)$ is easily implied. Jessen, Marcinkiewicz and Zygmund were the first to prove the *LlogL* inequality (*Note on the differentiability of multiple integrals, Funda. Math. 25 (1935), 217-34*), dominating M_s by iterates of the Hardy - Littlewood maximal operator M. However, the desired geometric proof was given 40 years later by A. Cordoba and R. Fefferman (*A geometric proof of the strong maximal theorem, Ann. of Math. 102 (1975), 95-100*).

Preliminaries

Basic definitions

- A differentiation basis $\mathcal{B} = \bigcup_{x\in\mathbb{R}^d} B(x)$, is a family of bounded measurable subsets of $\mathbb{R}^d$, with positive measure, such that $B(x)$ consists of sets that contain x and in $B(x)$ there are sets with arbitrary small diameter. $\mathcal{B}$ is called *Busseman-Feller*, if it consists of open sets and $x \in B \in \mathcal{B} \Rightarrow B \in B(x)$.
- If $\mathcal{B}$ is a differentiation basis, we define the associated maximal operator by $Mf(x) = \sup \frac{1}{|B|}\int_B |f|$, if $x \in \bigcup_{B\in B(x)} B$, $Mf(x) = 0$, otherwise, where the supremum is taken over all $B \in B(x)$.
- We define the upper and lower derivative of $\int f$ at x by $\overline{\mathcal{D}}(\int f, x) = \limsup \frac{1}{|B|}\int_B f$ and $\underline{\mathcal{D}}(\int f, x) = \liminf \frac{1}{|B|}\int_B f$ respectively, where the limits are taken over $B \in B(x)$ when $diam(B) \to 0$. If $\overline{\mathcal{D}}(\int f, x) = \underline{\mathcal{D}}(\int f, x) = f(x)$ for all $f \in X$ and for a.e. x, we say that the associated basis $\mathcal{B}$ differentiates X.
- We say that a sublinear operator T is of weak type (p,p), if $|\{Tf > \lambda\}| \lesssim (\frac{\|f\|_p}{\lambda})^p$, $\forall f \in L^p(\mathbb{R}^d)$, $\lambda > 0$. Also, we say that T satisfies the $LlogL$ inequality, if $|\{Tf > \lambda\}| \lesssim \int \frac{|f|}{\lambda}(1+\log^+\frac{|f|}{\lambda})^{d-1}$, $\forall f \in L^1_{loc}(\mathbb{R}^d)$, $\lambda > 0$. Finaly, we say that T is of strong type (p,p), if $\|Tf\|_p \lesssim \|f\|_p$, $\forall f \in L^p(\mathbb{R}^d)$.
- (**Strong Besicovitch covering property**) Let E be a measurable subset of $\mathbb{R}^d$. If to each $x \in E$ we correspond a set $S(x) \in B(x)$, then from the family $\{S(x)\}_{x\in E}$ one can obtain a sequence S_n satisfying:
  (1) $E \subset \bigcup_{n=1}^{\infty} S_n$
  (2) S_n can be distributed into c sequences, each of disjoint sets, where c is an absolute constant.
- (**Weak Besicovitch covering property**) Let E be a measurable subset of $\mathbb{R}^d$. If to each $x \in E$ we correspond a set $S(x) \in B(x)$, then from the family $\{S(x)\}_{x\in E}$ one can obtain a sequence S_n satisfying:
  (1) $E \subset \bigcup_{n=1}^{\infty} S_n$
  (2) $\sum_{n=1}^{\infty} \chi_{S_n} \leq c$, where c is absolute constant.
- (**Vitali covering property**) Let E be a measurable subset of $\mathbb{R}^d$. If to each $x \in E$ we correspond a sequence $Q_n(x) \in B(x)$ such that $diam Q_n(x) \to 0$, then from the family $\{Q_n(x)\}_{n\in\mathbb{N}, x\in E}$ one can obtain a sequence S_n of disjoint sets, satisfying $|E \setminus \bigcup_{n=1}^{\infty} S_n| = 0$.

Some general results

Let $\mathcal{B} = \bigcup_{x\in\mathbb{R}^d} B(x)$ be a differentiation basis and M be the associated maximal operator.

- M is of weak type (p,p), $p > 1 \implies \mathcal{B}$ differentiates $L^p(\mathbb{R}^d)$. If $\mathcal{B}$ is a B-F homothecy invariant basis, then the converse also holds.
- M is of weak type (p,p), $p > 1 \iff \mathcal{B}$ satisfies a q-type covering property (this is a Besicovitch-type covering property where the overlap is small in L^q-norm, $\frac{1}{p}+\frac{1}{q} = 1$).
- Vitali, w.B. and s.B. covering property $\implies$ $\mathcal{B}$ differentiates $L^1(\mathbb{R}^d)$. If $\mathcal{B}$ is a B-F homothecy invariant basis with $|\partial B| = 0$, $\forall B \in \mathcal{B}$, which differentiates $L^1(\mathbb{R}^d)$, then $\mathcal{B}$ satisfies the Vitali covering property.

The main theorem

Theorem 1 (Cordoba-Fefferman covering lemma for rectangles)**.** *Let $\{B_\alpha\}_{\alpha\in A}$ be a family of dyadic rectangles on $\mathbb{R}^d$, $d \geq 2$, such that $|\bigcup_{\alpha\in A} B_\alpha| < +\infty$. Then, there are $R_1, R_2, \dots, R_M \in \{B_\alpha\}_{\alpha\in A}$, satisfying:*

(1) $|\bigcup_{\alpha\in A} B_\alpha| \lesssim |\bigcup_{k=1}^{M} R_k|$

(2) $\int_{\bigcup_{k=1}^{M} R_k} e^{(\sum_{k=1}^{M}\chi_{R_k})^{\frac{1}{d-1}}} \lesssim |\bigcup_{k=1}^{M} R_k|$.

Sketch of proof

We will say that a sequence (finite or not) of rectangles R_n satisfies the property $\mathcal{P}_1$, if

$$|R_k \cap \bigcup_{j<k} R_j| \leq \frac{1}{2}|R_k|, \ \forall k,$$

whereas we will say that R_n satisfies the stronger property $\mathcal{P}_2$, if

$$|R_k \cap \bigcup_{j\neq k} R_j| \leq \frac{1}{2}|R_k|, \ \forall k.$$

The proof is by induction on dimension d. We work, equivalently, with the dyadic maximal operator M_d^d on $\mathbb{R}^d$, where the supremum is taken over all dyadic rectangles (cartesian products of dyadic intervals), in order to take advantage of their net *net property*, i.e. if I_1 and I_2 are dyadic intervals, then either they are disjoint, or one of them is contained in the other one.

The case $d = 2$

Lemma 1. *Let $B_1, B_2, \dots, B_N$ be dyadic rectangles on $\mathbb{R}^2$. Then one can choose $R_1, R_2, \dots, R_M \in \{B_j\}_{j=1}^N$, satisfying :*

(1) $|\bigcup_{j=1}^{N} B_j| \lesssim |\bigcup_{j=1}^{M} R_j|$

(2) $\mathcal{P}_2$ *property.*

Lemma 2. *Let $I_1, I_2, \dots, I_N \subset \mathbb{R}$ be discrete dyadic intervals, such that*

$$|\bigcup_{I_j \subsetneq I_k} I_j| \leq \frac{1}{2}|I_k|,$$

for each $k = 1, 2, \dots, N$. Also, set

$$A_r = \{x : \sum_{k=1}^{N} \chi_{I_k}(x) \geq r+1\},$$

$r = 0, 1, \dots$. Then,

$$|A_r| \lesssim \frac{1}{2^r}|\bigcup_{k=1}^{N} I_k|.$$

To prove **Theorem 1** on $\mathbb{R}^2$, fix $\{B_\alpha\}_{\alpha\in A}$. By **Lemma 1** there is $\{R_j\}_{j=1}^M \subset \{B_\alpha\}_{\alpha\in A}$, satisfying $\mathcal{P}_2$ property and *(1)* of Theorem 1. To prove *(2)* of Theorem 1, fix $x_2 \in \mathbb{R}$ and consider the x_2-sections $\{R_j^{x_2}\}_{j=1}^M$. These sections are dyadic intervals and by the net property one can show that they satisfy the hypotheses of **Lemma 2.**. Applying Lemma 2 to them, using Fubini's theorem and then integrating appropriately, we are done with the overlap condition *(2)*.

Induction step

Lemma 3. *Let T be a sublinear operator which acts on measurable functtions defined on $\mathbb{R}^{d-1}$, such that*

$$|x \in \mathbb{R}^{d-1} : Tf(x) > \lambda| \leq c \int_{\mathbb{R}^{d-1}} \frac{|f|}{\lambda}(1+\log^+\frac{|f|}{\lambda})^{d-2}$$

$$\&$$

$$\|Tf\|_\infty \leq \|f\|_\infty,$$

for all $f \in L^1_{loc}(\mathbb{R}^{d-1})$ and $\lambda > 0$, where c is an absolute constant. Then, for each $p > 1$, T is of weak type (p,p) with

$$|x \in \mathbb{R}^{d-1} : Tf(x) > \lambda| \leq \left(c2^{d+p-2}(\frac{d-2}{p-1})^{d-2}e^{p-1}\right)\frac{1}{\lambda^p}\int_{\mathbb{R}^{d-1}} |f|^p,$$

for all $f \in L^1_{loc}(\mathbb{R}^{d-1})$ and $\lambda > 0$.

Lemma 3, an interpolation argument and a duality argument yield the following auxiliary lemma.

Lemma 4. *Assume that the $(d-1)$-dimensional strong maximal operator M_s^{d-1} satisfies the LlogL inequality Then, given a finite family of rectangles $R_1, R_2, \dots, R_N \subset \mathbb{R}^{d-1}$ satisfying the $\mathcal{P}_1$ property, one has*

$$\int_{\bigcup_{k=1}^{N} R_k} e^{(\sum_{k=1}^{N}\chi_{R_k})^{\frac{1}{d-1}}} \lesssim |\bigcup_{k=1}^{N} R_k|.$$

Assume that **Theorem 1** is valid on $\mathbb{R}^{d-1}$. Applying **Lemma 1** to $\{B_\alpha\}_{\alpha\in A}$, ordering the R_j's appropriately, we extract $\{R_j\}_{j=1}^M \subset \{B_\alpha\}_{\alpha\in A}$ satisfying $\mathcal{P}_2$ property and *(1)* of Theorem 1. Applying Fubini's theorem and using the net property of the orthogonal projections $\{P_d(R_j)\}_{j=1}^M$ into d-axis (which clearly are dyadic intervals), one deduces that the *(d-1)*-dimensional dyadic rectangles $\{(R_j)_d^x\}_{j=1}^M$ ($(R)_d^x = \{\bar{x} \in \mathbb{R}^{d-1} : (\bar{x}, x) \in R)$ have the $\mathcal{P}_1$ property. Moreover, the induction hypothesis implies that M_d^{d-1} satisfies the assumptions of **Lemma 4**. Consequently, **Lemma 4** and an application of Fubini's theorem yield the overlap condition *(2)*, which completes the proof.

References

1. M. De Guzman. *Real Variable Methods in Fourier Analysis, North-Holland Math. Stud. 46, North-Holland, Amsterdam (1981).*
2. M. De Guzman. *Differentiation of Integrals in $\mathbb{R}^n$, Lecture Notes in Math. 481, Springer (1975).*
3. A. Cordoba and R. Fefferman. *A geometric proof of the strong maximal theorem, Ann. of Math. 102 (1975), 95-100.*
4. A. Cordoba. *On the Vitali covering properties of a differentiation basis, Studia Math. 57 (1976), 91-95.*
5. J. Duoandikoetxea. *Fourier Analysis, American Mathematical Society, (2001).*
6. T. Mitsis. *The weighted weak type inequality for the strong maximal function, J. Fourier Anal. Appl. 12 (2006), 645-652.*
7. A. A. Almirall, C. C. Gomez, H. J. Sanchez, C. M. Murillo, A. Quilis and P. R. Baraut. *Maximal averaging operators: from geometry to boundedness through duality (2019).*