A Tour on Spectral Graph Theory and Expander Graphs

Master's Thesis

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Abstract

In the first part of this thesis we study the tools from spectral graph theory and some graph theoretic results that are obtained through the eigenvalues of the Laplacian and Adjacency matrix of a graph. Furthermore we investigate the eigenvalues of different families of graphs such as: Cayley graphs of groups, binary trees and paths but also the convergence rate of random walks on graphs. In the second part we focus on a specific class of graphs that is called Expander graphs. We prove the existence of the family of magical graphs with high probability and their applications to probability theory, random walks and complexity theory.

1. Spectral Graph Theory

The Adjacency matrix of a graph G with n vertices can be represented as a symmetric $n \times n$ matrix where the non-diagonal elements a_{ij} represents the number of edges from vertex i to vertex j

Definition 1.1 (Adjacency matrix [2]) *The adjacency matrix M_G of a graph $G = (V, E)$, whose entries $M_G(a, b)$ are given by:*

$$M_G(a,b) = \begin{cases} 1 & \text{if } (a,b) \in E \\ 0 & \text{otherwise.} \end{cases}$$

In case G is a weighted graph with an edge (a, b) having weight $w_{a,b}$, we set $M_G(a, b) = w_{a,b}$, also an important thing to notice is that we index the rows and columns of the matrix by vertices instead of numbers.

Definition 1.2 *(Laplacian Matrix) The Laplacian Matrix L_G of a graph $G = (V, E)$ is defined as: $L_G = D_G - M_G$, where D_G, M_G are the degree matrix and the adjacency matrix of the graph G respectively. We can also define the entries $L_G(i, j)$ as:*

$$L_G(i,j) = \begin{cases} deg(v_i) & \text{if } i = j \\ -1 & \text{if } (i,j) \in E \\ 0 & \text{otherwise} \end{cases}$$

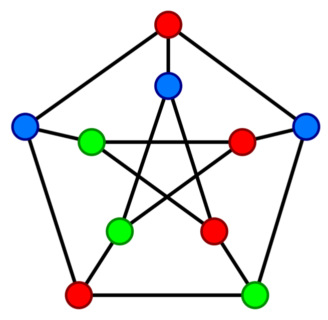
Lemma 1.3 *Let $G = (V, E, w)$ be a weighted graph, and let $0 = \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$ be the eigenvalues of its Laplacian matrix $\boldsymbol{L}$. Then we have that $\lambda_2 > 0 \iff G$ is connected.*

Lemma 1.4 *For a weighted graph G with n vertices, we define:*

$$d_{ave} = \frac{\sum_a \boldsymbol{d}(a)}{n} \text{ and } d_{max} = \max_a \boldsymbol{d}(a).$$

A graph is called also k-colorable if it can be colored using only k colors, usually we often identify these k colors with integers from the set $\{1, \ldots, k\}$, so a k-coloring of a graph is a function $f : \{1, 2, \ldots, k\} \to V(G)$ such that $f(a) \neq f(b)$ for all $(a, b) \in E$.

We can see also below a graph that is 3-colorable.



Theorem 1.5 *(Wilf's Theorem)*

$$\chi(G) \leq \lfloor \mu_1 \rfloor + 1$$

2. Generalized hypercubes

To generalize the hypercube, we will now consider again the $Cay(\Gamma, S)$ over the additive group $\Gamma = (\mathbb{Z}/2\mathbb{Z})^d$ as before but with a different set of generators in contrast to the definition of hypercube. The vertex set of the graph will be $V_\Gamma = \{0,1\}^d \mod 2$.

Each generator now $g_1, g_2, \ldots, g_k$ belongs to the same group. As $g + g = 0 \; mod \; 2$ for all generators $g \in \{0,1\}^d$, we can observe that the set of generators is closed under negation. Let now define G to be the Cayley graph with vertex set $V_G = \{0,1\}^d$ and with edge set $E_G = \{x, x + g_j, \; x \in V_G, \; 1 \leq j \leq k\}$. For each $b \in \{0,1\}^d$, we can define a function $\psi_b : V_G \to \mathbb{R}$ such that $\psi_b(x) = (-1)^{b^T x}$.

Lemma 2.1 *For every $b \in \{0,1\}^d$ the vector ψ_b is a Laplacian matrix eigenvector corresponds to the eigenvalue:*

$$k - \sum_{i=1}^{k} (-1)^{b^T g_i}$$

But now what can we say if the set of generators is random an with random means to choose the set of generators $\{g_1, g_2, \ldots, g_k\}$ uniformly at random, where $k = cd, c > 0$ is a multiple constant of the dimension, then we can obtain a good approximation of the complete graph.

Theorem 2.2 *Let $\{x_1, x_2, \ldots, x_k\}$ be independent $\{\pm 1\}$ random variables with zero mean. Then for all $t > 0$, we have:*

$$P(|k - \lambda_b| \geq t) = P(|k - (k - \sum_{i=1}^{k} (-1)^{b^T g_i})| \geq t) =$$
$$P(\sum_{i=1}^{k} (-1)^{b^T g_i})| \geq t) = P(|\sum_i x_i| \geq t) \leq 2 \cdot e^{-\frac{t^2}{2k}}.$$

Theorem 2.3 *With high probability, all the nonzero eigenvalues of the generalized hypercube differ from k by at most:*

$$k\sqrt{\frac{2}{c}}$$

where $k = cd$

3. Inequalities for eigenvalues of graphs

Lemma 3.1 *Consider the following approximation for graphs, $(n-1) \cdot P_n \succeq G_{1,n}$, where P_n is the path graph from vertex 1 to n and $G_{1,n}$ is the graph with just one edge $(1, n)$.*

A lower bound on the complete binary tree:

$$\lambda_2(T_n) \geq \frac{1}{(n-1) \cdot log_2(n)}$$

There is also an improved version of the previous inequality:

Lemma 3.2

$$\lambda_2(T_n) \geq \frac{1}{2n}$$

4. Expander graphs

Expander graphs[1] have been widely used to solve problems in different mathematical domains such as number theory and Complexity theory. To begin our exploration into expander graphs we will state a problem that its solution need a special class of expander graphs that is called magical graphs.

Let $\mathbb{F}$ be a finite field and A to be a linear transformation over $\mathbb{F}$. The question now is that we would like to build a circuit which computes the transformation $x \to Ax$. Each gate of the circuit computes addition or multiplication.

Conjecture 4.1 *(Valiant's Conjecture) Every super concentrator graph must have greater than n edges and this implies that every circuit that computes a super regular matrix must have greater than n gates*

Let $G = (L, R, E)$ to be a two-sided bipartite graph on n vertices on each side (n on the left side and n on the right side). Lets denote with L to be the set of all vertices that are belong to the left side and similarly with R to be the vertices of the right side, also we assume that every left vertex in L has d neighbors on the right side R so every vertex on the left side has the same degree that is d.

We will name that G, to be a (d, n)-magical graph if it has the following properties:

- For every subset of vertices $S \subseteq L$ such that $|S| \leq \frac{n}{3d} \implies |\Gamma(S)| \geq |S| \cdot \frac{d}{4}$
- For every subset of vertices $S \subseteq L$ such that $\frac{n}{3d} < |S| \leq \frac{n}{2} \implies |\Gamma(S)| \geq |S| + \frac{n}{3d}$

with $\Gamma(S)$ we denote the set of neighbors of vertices in S in the graph G.

Lemma 4.2 *For every $d \geq 8$ and sufficiently large n, there exists a (d, n) magical graph.*

Lemma 4.3 *The expander mixing lemma For every subset $S, T \subseteq V$ it holds:*

$$||E(S,T) - \frac{d|S||T|}{n}| \leq \lambda \cdot \sqrt{|S| \cdot |T|}$$

Theorem 4.4 *(Alon-Boppana Bound) In every d-regular graph G it holds that:*

$$\lambda \geq 2\sqrt{d-1} - o_n(1)$$

References

[1] Shlomo Hoory, Nathan Linial, and Avi Wigderson. "Expander graphs and their applications". In: *Bulletin of the American Mathematical Society* 43.4 (2006), pp. 439–561.

[2] Daniel Spielman. "Spectral and algebraic graph theory". In: *Yale lecture notes, draft of December* 4 (2019), p. 47.