Hausdorff Dimension and Energy of Measures

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Abstract
The aim of the project is to prove Marstrand's Projection Theorem for one dimensional projections.J.Marstrand proved, in 1954 that: If $E \subseteq \mathbb{R}^2$ is a Borel set with Hausdorff dimension $s \leq 1$, then the Hausdorff dimension of the projection of E on almost every line L through the origin has again Hausdorff dimension s, otherwise if $s > 1$ then E projects into a set of positive length in almost all directions.In this thesis,we give R.Kauffman's proof(1968) of the theorem in $\mathbb{R}^d$. In his proof, R.Kauffman makes natural use of the potential theoretic characterization of Hausdorff dimension and Fourier transform methods. Here we shall prove the theorem only for the projections of compact subsets of $\mathbb{R}^d$.

Introduction

Given a Borel set $E \subseteq \mathbb{R}^d$ and $a, \delta > 0$ we define $H_a^\delta(E) = \inf\left(\sum_j r_j^a\right)$ where the infimum is taken over all countable coverings of E by balls $B(x_j, r_j)$ with $r_j < \delta$. It is clear that $H_a^\delta(E)$ increases as δ decreases, and we define the $a-$Hausdorff measure of E the quantity $H_a(E) = \lim_{\delta \to 0} H_a^\delta(E)$ It is also clear that $H_a^\delta(E) \leq H_\beta^\delta(E)$ if $a > \beta$ and $\delta \leq 1$, thus $H_a(E)$ is a nonincreasing function of a. We also denote $H_a^\infty(\cdot)$ the set function $H_a^\delta(\cdot)$ when $\delta = +\infty$.
H_a is a metric outer measure on $\mathbb{R}^d$, and hence, all Borel sets in $\mathbb{R}^d$ are H_a-measurable.There is a unique number a_0, called the Hausdorff dimension of E or $\mathbf{dim}(E)$, such that $H_a(E) = \infty$ if $a < a_0$ and $H_a(E) = 0$ if $a > a_0$.It is proved that $\mathbf{dim}(E) = \sup\{a \geq 0 : H_a(E) = +\infty\} = \inf\{a \geq 0 : H_a(E) = 0\}$.
In this thesis,in chapter II, III we state and prove some theorems from measure theory and harmonic analysis which will be used later.In chapters IV and V we describe the notion of Hausdorff dimension in terms of energies,and the connection between the Riesz Energy and Fourier Transform.In chapter VI we state and prove the main theorem and also a stronger result about compact sets of dimension greater than 2.This thesis is based on the lecture notes taught by Tom Wolff in the Spring of 2000 at the California Institute of Technology,the lectures notes of Tuomas Orponen for the course Geometric measure theory, given at the University of Helsinki in 2018 and P.Mattila's book,"Fourier Analysis and Hausdorff Dimension".

Frostman's Lemma

Definition:A Borel measure μ on $\mathbb{R}^d$ is called an **s-Frostman** measure, if there is a constant $C > 0$, such that $\mu(B(x, r)) \leq Cr^s, \forall x \in \mathbb{R}^d, \forall r > 0$.

Frostman's Lemma: Assume that $E \subseteq \mathbb{R}^d$ is a compact. Then $H_s(E) > 0$ if and only if there exists a non-zero $s-$Frostman measure μ with $supp(\mu) \subseteq E$.

Riesz Energy

We now define the $a-$**dimensional Riesz energy** of a (positive) measure μ with compact support(the compact support assumption is not needed; it is included to simplify the presentation) by the formula

$$\mathbb{I}_a(\mu) = \int_{\mathbb{R}} \int_{\mathbb{R}} |x - y|^{-a} d\mu(x) d\mu(y)$$

We always assume that $0 < a < d$ and we also define the **Riesz potential**

$$\mathbb{V}_\mu^a(x) = \int_{\mathbb{R}} |x - y|^{-a} d\mu(y)$$

(which is the convolution of $|x|^{-a}$ with μ). Thus

$$\mathbb{I}_a(\mu) = \int_{\mathbb{R}} \mathbb{V}_\mu^a d\mu$$

Lemma: (i) If μ is a probability $a-$Frostman measure with compact support then $\mathbb{I}_\beta(\mu) < \infty$, $\forall \beta < a$.
(ii) Conversely,if μ is a probability measure with compact support and with $\mathbb{I}_a(\mu) < \infty$ then there exists a probability $a-$Frostman measure ν such that $\nu(X) \leq 2\mu(X)$ for all sets $X \subseteq \mathbb{R}^d$

If $E \subseteq \mathbb{R}^d$ we denote $P(E)$ the set of probability Borel measures supported on E.

Theorem: If $E \subseteq \mathbb{R}^d$ compact,then the Hausdorff dimension of E coincides with the number

$$\sup\{a : \exists \mu \in P(E) \text{ with } \mathbb{I}_a(\mu) < \infty\}$$

Riesz Energy and Fourier Transform

The **Fourier Transform** of a complex Borel measure μ on $\mathbb{R}^d$ is given by:

$$\hat{\mu}(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i \xi \cdot x} d\mu(x)$$

The connection between Riesz Energy and Fourier transform is given by the following theorem.

Theorem: Let μ be a positive measure with compact support
and $0 < a < d$. Then

$$\mathbb{I}_a(\mu) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |x - y|^{-a} d\mu(x) d\mu(y) = c_a \int_{\mathbb{R}^d} |\hat{\mu}(\xi)|^2 |\xi|^{-(d-a)} d\xi$$

where $c_a = \frac{\Gamma(\frac{d-a}{2})\pi^{a-\frac{d}{2}}}{\Gamma(\frac{a}{2})}$

Main Theorems

For $e \in \mathbb{S}^{d-1}, d \geq 2$, define the projection $P_e : \mathbb{R}^d \to \mathbb{R}$,

$$P_e(x) = e \cdot x$$

This is essentially the orthogonal projection onto the line $L = \{te : t \in \mathbb{R}\}$.

If μ is a measure supported on a compact set E and $e \in \mathbb{S}^{d-1}$ then we denote μ_e the projected measure $P_e\mu(B) = \mu(P_e^{-1}(B)),\ B \subseteq \mathbb{R}$.

With "a.e. $e \in \mathbb{S}^{d-1}$" we always mean, almost everywhere with respect to the surface measure σ on $\mathbb{S}^{d-1}$.

Theorem (Marstrand) Assume that $E \subseteq \mathbb{R}$ is compact and $\mathbf{dim}(E) = a$. Then: (i) If $a \leq 1$ then for a.e. $e \in \mathbb{S}^{d-1}$ we have $\mathbf{dim}(P_e(E)) = a$.
(ii) If $a > 1$ then for a.e. $e \in \mathbb{S}^{d-1}$ we have $m_1(P_e(E)) > 0$.

Theorem: Let $E \subseteq \mathbb{R}$ compact with $\mathbf{dim}(E) > 2$.
Then $P_e(E)$ has non-empty interior for a.e. $e \in \mathbb{S}^{d-1}$.

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References

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