WIGNERIZATION OF THE HELMHOLTZ EQUATION WITH POINT SOURCE

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Introduction

In this poster we summarize the results of a graduate thesis dealing with the solution of the Helmholtz equation with a point source in an inhomogeneous medium with linear refraction index. Using the path-linking formulation, we relate the Helmholtz equation to an initial value problem for a Schrödinger equation and construct its wavefunction via the Wigner transform. The resulting Green function, corresponding to the Airy equation, is compared to the one obtained by standard methods. Motivated by Hannays interpretation of path linking and its similarity to Feynmans path integrals, we conjecture that Hannay's paths are the images of Feynman's paths, and that time integration of Feynman's integral should provide the Green function of the Helmholtz equation.

The Helmholtz equation

We consider the 1-D inhomogeneous wave equation

$$c^2(x)\frac{\partial^2 Y}{\partial x^2} = \frac{\partial^2 Y}{\partial t^2} + F(x,t)\ ,\ x\in\mathbb{R}\ ,t\geq 0\ ,$$

where the external force is time-harmonic with a fixed frequency ω

$$F(x,t) = f(x)e^{-i\omega t}\ .$$

By seeking a time-harmonic solution of the wave field Y

$$Y(x,t) = u(x)e^{-i\omega t}\ ,$$

we obtain the inhomogeneous Helmholtz-type equation

$$u''(x)+k_0^2\eta(x)u(x) = \frac{1}{c^2(x)}f(x)\ ,\ k_0 = \omega/c_0\ ,\ \eta(x) = \left(\frac{c_0}{c(x)}\right)^2$$

which can be rescaled by using a characteristic length L

$$\epsilon^2 u_\epsilon''(x) + \eta(x)u_\epsilon(x) = \sigma_\epsilon\delta(x-x_0)\ ,\quad \epsilon = \frac{1}{k_0 L} << 1\ .$$

The basic example: Airy equation

We will consider the case of linear refraction index $\eta(x) = x$ and with a change of variables

$$z = -\frac{x}{\epsilon^n},\quad w(z) = u_\epsilon(x)\ ,$$

it turns out that w satisfies the inhomogeneous Airy equation

$$w''(z) - zw(z) = \sigma_\epsilon\epsilon^{-4/3}\delta(z-z_0)\ .$$

The linearly independent fundamental solutions of the homogeneous Airy equation are

$$\begin{cases} w_1(z) = Ai(z) \\ w_2(z) = Bi(z)\end{cases}$$

where $Ai(z)$ is the Airy function of the first kind and $Bi(z)$ of the second kind.

The Green's function has the representation

$$w(z) = \begin{cases} aA_i(z) + bB_i(z)\ ,\ z<z_0 \\ cA_i(z) + dB_i(z)\ ,\ z>z_0\end{cases}$$

By its definition it must be bounded and satisfy the continuity and jump conditions

$$\begin{cases} w(z_0+0) = w(z_0-0) \\ w'(z_0+0) - w'(z_0-0) = 1\end{cases}$$

We must also impose Sommerfeld-type radiation conditions as $z\to\pm\infty$, which require the solution to represent outgoing wave behavior. Finally we obtain that

$$u_\epsilon(x) = -i\pi\sigma_\epsilon\epsilon^{-4/3}\begin{cases} A_i(-x_0\epsilon^{-\frac{2}{3}})(Ai(-x\epsilon^{-\frac{2}{3}}) - iB_i(-x\epsilon^{-\frac{2}{3}})), x>x_0 \\ Ai(-x\epsilon^{-\frac{2}{3}})(Ai(-x_0\epsilon^{-\frac{2}{3}}) - iBi(-x_0\epsilon^{-\frac{2}{3}})), x<x_0\end{cases}$$

In order $u_\epsilon(x) = O(1)$ with respect to ϵ, as $x\to\infty$, we have to chose $\sigma_\epsilon = -i\epsilon e^{i\frac{\pi}{4}}$.

Path linking formulation of the Helmholtz equation

We consider the path linking formulation

$$u_\epsilon(x) = i\int_0^\infty U^\epsilon(x,t)dt\ ,$$

where we choose $U^\epsilon(x,t)$ to satisfy the time independent Schrödinger equation

$$i\epsilon\frac{\partial U^\epsilon}{\partial t} = -\frac{\epsilon^2}{2}\frac{\partial^2 U^\epsilon}{\partial x^2} - \frac{1}{2}\eta(x)U^\epsilon\ ,$$

and the initial data are related to the point source excitation in the Helmholtz equation such that

$$U^\epsilon(x,t=0) = U_0^\epsilon(x) = -\frac{\sigma_\epsilon}{2\epsilon}\delta(x-x_0)\ .$$

Therefore, $U^\epsilon(x,t)$ must be the solution of the IVP

$$i\epsilon\frac{\partial U^\epsilon}{\partial t} = -\frac{\epsilon^2}{2}\frac{\partial^2 U^\epsilon}{\partial x^2} + V(x)U^\epsilon\quad ,\ V(x) = -\frac{\eta(x)}{2},\quad U^\epsilon(x,t=0) = U_0^\epsilon(x)$$

so that the path linking formulation is a solution of the Helmholtz equation.

The Wigner transform and the Wigner equation

We define the *Wigner transform* $W[f](x,k)$ of a complex-valued function $f(x)$ as

$$W[f](x,k) := \frac{1}{2\pi}\int_{-\infty}^\infty e^{-ik\zeta}f(x+\frac{\zeta}{2})\overline{f}(x-\frac{\zeta}{2})d\zeta$$

and the corresponding *scaled Wigner transform* as

$$W^\epsilon[f](x,k) := \frac{1}{\epsilon}W[f]\left(x,\frac{k}{\epsilon}\right) = \frac{1}{\pi\epsilon}\int_{-\infty}^\infty e^{-i\frac{2k}{\epsilon}\sigma}f(x+\sigma)\overline{f}(x-\sigma)d\sigma\ .$$

For the Schrödinger equation

$$i\epsilon\partial_t U^\epsilon(x,t) = -\frac{\epsilon^2}{2}\partial_x^2 U^\epsilon(x,t) + V(x)U^\epsilon(x,t)\ ,\ x\in\mathbb{R}\ ,\ t>0$$

when the potential $V(x)$ is smooth, it turns out that $W^\epsilon(x,k,t) = W^\epsilon[U^\epsilon](x,k,t)$ satisfies the *Wigner equation*

$$\partial_t W^\epsilon + k\partial_x W^\epsilon - V'(x)\partial_k W^\epsilon = \sum_{m=1}^\infty \alpha_m\epsilon^{2m}V^{(2m+1)}(x)\partial_k^{(2m+1)}W^\epsilon$$

where $\alpha_m = (-1)^m 2^{-2m}/(2m+1)!\ ,\ \ m = 1,2,\ldots$
Also the the Wigner transform of the Dirac delta function $\delta_{x_0}(x)\equiv\delta(x-x_0)$ is equal to

$$W^\epsilon[\delta_{x_0}](x,k) = \frac{1}{2\pi\epsilon}\delta(x-x_0)\ .$$

Wigner equation for the linear potential

We reform the IVP with a linear potential $V(x) = -\frac{\eta(x)}{2} = -\frac{x}{2}$ into an initial value problem for the Wigner transform $W^\epsilon = W^\epsilon[U^\epsilon]$ as follows

$$\begin{cases} \partial_t W^\epsilon + k\partial_x W^\epsilon + \frac{1}{2}\partial_k W^\epsilon = 0\ , \\ W^\epsilon(x,k,t=0) = W_0^\epsilon(x,k)\equiv\frac{1}{8\pi\epsilon}\delta(x-x_0)\ .\end{cases}$$

which is a simple transport equation and

$$W^\epsilon(x,k,t) = W_0^\epsilon(q(x,k,t),p(x,k,t)) = \frac{1}{8\pi\epsilon t}\delta\left(k-\frac{t}{4}-\frac{x-x_0}{t}\right)\ .$$

We derive the wavefunction $U^\epsilon(x,t) = a_\epsilon(x,t)e^{i\phi_\epsilon(x,t)}$, by using the properties of the Wigner transform

$$\int_{-\infty}^\infty W^\epsilon(x,k)dk = |U^\epsilon(x)|^2\ ,$$

$$\int_{-\infty}^\infty kW^\epsilon dk = -\frac{i\epsilon}{2}\left(\overline{U^\epsilon}(x)\,\partial_x U^\epsilon(x)\ - U^\epsilon(x)\,\partial_x\overline{U^\epsilon}(x)\right)$$

and thus

$$U^\epsilon(x,t) = i\left(8\pi\epsilon t\right)^{-\frac{1}{2}}exp\left(\frac{i}{\epsilon}\left(\frac{(x-x_0)^2}{2t}+\frac{(x+x_0)t}{4}-\frac{t^3}{96}\right)\right)\ .$$

We can now return to the path linking formulation and construct the solution u_ϵ of the Helmholtz equation

$$u_\epsilon(x) = i\int_0^\infty U^\epsilon(x,t)dt = -\pi e^{i\pi/4}\epsilon Ai(u)\left(Ai(v)-iBi(v)\right)\epsilon^{-4/3}$$

where

$$\left(u = -\frac{x_0}{\epsilon^{2/3}}\ ,\ v = -\frac{x}{\epsilon^{2/3}}\right)\ ,x>x_0$$

or

$$\left(u = -\frac{x}{\epsilon^{2/3}}\ ,\ v = -\frac{x_0}{\epsilon^{2/3}}\right)\ ,x<x_0\ .$$

which coincides with the Green's function (with the chosen normalization factor $\sigma_\epsilon = -i\epsilon e^{i\frac{\pi}{4}}$).

The Feynman path integral

In quantum mechanics, a particle does not follow a single classical trajectory. Instead, it explores all possible paths connecting two points in space-time. The probability amplitude G satisfies the Schrödinger equation

$$i\hbar\frac{\partial G}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 G}{\partial x^2} + V(x)G\ ,\quad G(x,0) = \delta(x-y)\ .$$

Let a quantum particle of mass m move from position y at time 0 to position x at time t. Denote by $\mathcal{P}_{x,y}^{t,0}$ the set of *all* continuous paths $z(\tau)$ such that $z(0) = y$ and $z(t) = x$.
The probability amplitude for the particle to propagate from y to x in time t is then formally expressed as the path integral

$$G(x,y,t) = \int_{\mathcal{P}_{x,y}^{t,0}}\exp\left(\frac{i}{\hbar}S[z(\cdot),t]\right)\mathcal{D}z(\cdot)\ .$$

For the *linear potential* $V(x) = \alpha x\ ,\alpha = \mathrm{const.}$, we derive

$$G_l(x,y,t) = \sqrt{\frac{m}{2\pi i\hbar t}}\exp\left(\frac{i}{\hbar}\Big(\frac{m(x-y)^2}{2t}-\frac{\alpha(x+y)t}{2}-\frac{\alpha^2t^3}{24m}\Big)\right)$$

For our case $m=1,\hbar=\epsilon,\alpha=-1/2$, with initial data $U_0^\epsilon(x) = \frac{ie^{i\frac{\pi}{4}}}{2}\delta(x-x_0)$. By using Duhamel's formula the solution to the initial value problem is

$$\begin{aligned} U^\epsilon(x,t) &= \int_{-\infty}^\infty G_l(x,y,t)U_0^\epsilon(y)\,dy \\ &= i\left(8\pi\epsilon t\right)^{-\frac{1}{2}}exp\left(\frac{i}{\epsilon}\left(\frac{(x-x_0)^2}{2t}+\frac{(x+x_0)t}{4}-\frac{t^3}{96}\right)\right)\ .\end{aligned}$$

Therefore, by "using the path integral" of the propagator Green's function of the Schrödinger equation, we found the formula we constructed through the Wigner transform.

Conjecture: Hannay's path links are the images of Feynman's paths in the set $\mathcal{P}_{x,y}^{t,0}$, as t goes to ∞.

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