Bergman Spaces

Master Thesis

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Summary

We study the functions of the Bergman spaces $\mathcal{A}^p$, that is, the holomorphic functions on the unit disc that are also p-integrable with respect to the Lebesgue area measure. In particular, we explore the properties of these spaces, such as their completeness and duals, obtaining results that are analogous to the properties of L^p spaces. Also, we present a connection between the Bergman spaces and the hyperbolic metric. More precisely, we show that each function of the Bergman space is an infinite sum that depends on specific sequences of the hyperbolic disc. Finally, we distinguish the case $p = 1$, for which we introduce the Bloch space as the dual of $\mathcal{A}^1$.

Hyperbolic geometry on the unit disc

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. We will introduce a metric on $\mathbb{D}$, called hyperbolic metric of the unit disc.

Definition 1. *If $\gamma : [a, b] \to \mathbb{D}$ is a piecewise-C^1 curve, we define the hyperbolic length of γ to be the real number*

$$l_h(\gamma) = \int_a^b \frac{|\gamma'(t)|}{1 - |\gamma(t)|^2}\,dt.$$

Definition 2. *If $z, w \in \mathbb{D}$, we define the hyperbolic distance of z, w to be the number*

$$\beta(z, w) = \inf\{l_h(\gamma) : \gamma \text{ is a piecewise-}C^1 \text{ curve in } \mathbb{D} \text{ starting at } z \text{ and landing at } w\}.$$

Then, β is a metric on $\mathbb{D}$, and we can find an explicit formula for it, namely,

$$\beta(z, w) = \frac{1}{2} \log \frac{1 + \left|\frac{z - w}{1 - \bar{w}z}\right|}{1 - \left|\frac{z - w}{1 - \bar{w}z}\right|}, \text{ for all } z, w \in \mathbb{D}.$$

Also, $(\mathbb{D}, \beta)$ is a complete metric space.

Notation. If $z \in \mathbb{D}$ and $r > 0$, we denote by $D_h(z, r)$ the hyperbolic disc with center z and radius r, that is, $D_h(z, r) := \{w \in \mathbb{D} : \beta(w, z) < r\}$.

Now we introduce the notion of r-lattices, which will be found useful later.

Definition 3. *Let $r > 0$. A sequence $\{a_k\}$ in $\mathbb{D}$ is called an r-lattice in the hyperbolic metric if:*
(i) $\beta(a_i, a_j) \geq \frac{r}{2}$, for all $i \neq j$, and
(ii) $\mathbb{D} = \bigcup_{k=1}^{+\infty} D_h(a_k, r)$.

These sequences induce partitions of the unit disc:

Proposition 1. *Suppose $0 < r \leq 1$ and let $\{a_k\}$ be an r-lattice in the hyperbolic metric. Then, for each $k \in \mathbb{N}$ there exists a Borel-measurable set $D_k \subset \mathbb{D}$ with the following properties:*
(i) $D_h(a_k, \frac{r}{4}) \subset D_k \subset D_h(a_k, r)$, for all $k \in \mathbb{N}$,
(ii) $D_i \cap D_j = \emptyset$, if $i \neq j$, and
(iii) $\mathbb{D} = \bigcup_{k=1}^{+\infty} D_k$.

Properties of the Bergman Spaces

Let μ_2 denote the restriction to $\mathbb{D}$ of the Lebesgue measure of $\mathbb{C}$. Then, by A we denote the normalized area measure of $\mathbb{D}$, that is $A = \frac{1}{\pi}\mu_2$. This way, $A(\mathbb{D}) = 1$.

Definition 4. *For $p > 0$ we define $\mathcal{A}^p(dA) = H(\mathbb{D}) \cap L^p(\mathbb{D}, dA)$. These spaces are called Bergman spaces.*

Remark 1. *The Bergman spaces are vector spaces over $\mathbb{C}$, as $H(\mathbb{D})$ and $L^p(\mathbb{D}, dA)$ are vector spaces over $\mathbb{C}$.*

Theorem 1. *If $p \geq 1$, then the space $(\mathcal{A}^p(dA), \|\cdot\|_p)$, where*

$$\|f\|_p = \left(\int_{\mathbb{D}} |f|^p\,dA\right)^{1/p}, \text{ for all } f \in \mathcal{A}^p(dA),$$

is a Banach space.
If $0 < p < 1$, then the space $(\mathcal{A}^p(dA), d_p)$, where

$$d_p(f, g) = \int_{\mathbb{D}} |f - g|^p\,dA, \text{ for all } f, g \in \mathcal{A}^p(dA),$$

is a complete metric space.

Proposition 2. *Let $p \geq 1$ and $z \in \mathbb{D}$. The function $T_z : \mathcal{A}^p(dA) \to \mathbb{C}$, with $T_z(f) = f(z)$, for all $f \in \mathcal{A}^p(dA)$, is a bounded linear functional on $\mathcal{A}^p(dA)$.*

It is known from Functional Analysis that the Banach space $L^2(\mathbb{D}, dA)$ can be supplied with the inner product

$$\langle f, g\rangle_2 = \int_{\mathbb{D}} f(w)\overline{g(w)}\,dA(w), \text{ for all } f, g \in L^2(\mathbb{D}, dA),$$

and that $\|\cdot\|_2$ is induced by this inner product. This fact makes $(L^2(\mathbb{D}, dA), \langle\cdot,\cdot\rangle_2)$ a Hilbert space, so $(\mathcal{A}^2(dA), \langle\cdot,\cdot\rangle_2)$ is also a Hilbert space.
Let $z \in \mathbb{D}$. Since $T_z \in (\mathcal{A}^2(dA))^*$, by the Riesz representation theorem there exists a unique function $h_z \in \mathcal{A}^2(dA)$ such that for all $f \in \mathcal{A}^2(dA)$,

$$T_z(f) = \langle f, h_z\rangle_2 \;\Leftrightarrow\; f(z) = \int_{\mathbb{D}} f(w)\overline{h_z(w)}\,dA(w).$$

Let $K : \mathbb{D} \times \mathbb{D} \to \mathbb{C}$ with $K(z, w) = \overline{h_z(w)}$, for all $z, w \in \mathbb{D}$. K is called the Bergman kernel of $\mathbb{D}$, and we can find an explicit formula for it:

$$K(z, w) = \frac{1}{(1 - z\bar{w})^2}, \text{ for all } z, w \in \mathbb{D}.$$

In fact, it holds that

$$f(z) = \int_{\mathbb{D}} f(w)K(z, w)\,dA(w), \text{ for all } z \in \mathbb{D} \text{ and } f \in \mathcal{A}^p(dA),$$

where $p \geq 1$.

Theorem 2. *Let $1 < p < +\infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then $(\mathcal{A}^p(dA))^*$ and $\mathcal{A}^q(dA)$ are isomorphic, with equivalent norms.*

We now present the atomic decomposition that holds in Bergman spaces.

Theorem 3. *Let $p \geq 1$ and $b > 2 \geq 1 + \frac{1}{p}$. Then there exists a constant $\sigma = \sigma(p, b) > 0$ such that for any r-lattice $\{a_k\}$ in the hyperbolic metric, where $0 < r < \sigma$, the space $\mathcal{A}^p(dA)$ consists exactly of functions of the form*

$$f(z) = \sum_{k=1}^{+\infty} c_k \frac{(1 - |a_k|^2)^{(pb-2)/p}}{(1 - z\bar{a}_k)^b}, \text{ for all } z \in \mathbb{D}, \tag{1}$$

where $\{c_k\} \in l^p$, and the series (1) converges in norm in $\mathcal{A}^p(dA)$.

The proof of this theorem uses a variety of tools, including Proposition 1.

The Bloch Space

Our goal is to find the dual of $\mathcal{A}^1(dA)$. Let $f \in H(\mathbb{D})$ and set $\|f\|_{\mathcal{B}} = \sup\{(1 - |z|^2)|f'(z)| : z \in \mathbb{D}\}$. We define the Bloch space $\mathcal{B}$ of $\mathbb{D}$ to be the space of all analytic functions f on $\mathbb{D}$ such that $\|f\|_{\mathcal{B}} < +\infty$.

Remark 2. *The Bloch space is a vector space over $\mathbb{C}$.*

We now introduce the norm $\|f\| = |f(0)| + \|f\|_{\mathcal{B}}$ in $\mathcal{B}$.

Proposition 3. *$(\mathcal{B}, \|\cdot\|)$ is a Banach space.*

Theorem 4. *$(\mathcal{A}^1(dA))^*$ and $\mathcal{B}$ are isomorphic, with equivalent norms.*

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