Graduate Course Catalogue

M.Sc. in Mathematics and its Applications

The following section describes the graduate courses offered in the program. These are formal courses consisting of weekly lectures, problem sets, assignments, and examinations (midterms and finals). Course content may vary slightly from year to year to reflect the research interests of the faculty and students. Specifically, the curriculum for "Topics" courses is proposed by the instructor and approved by the Graduate Committee. All course descriptions are subject to change by the department's governing bodies.

Section A: Algebra, Logic, and Discrete Mathematics

A10: ALGEBRA II

- **I. Groups:** Group actions on sets. Sylow theorems. Properties of p-groups. Nilpotent groups. Solvable groups. Jordan-Hölder theorem.
- **II. Rings:** Rings, subrings, ideals, prime and maximal ideals. Associative, prime, and irreducible elements. Principal Ideal Domains (PID), Unique Factorization Domains (UFD), Euclidean Domains, and their relationships. Noetherian and Artinian rings. Polynomial rings. Hilbert's Basis Theorem and Nullstellensatz. Gauss's Theorem. Primary ideals and the Lasker-Noether theorem. Valuation rings and Dedekind domains.
- **III. Modules:** Modules, homomorphisms, and exact sequences. Free modules and basic properties. Projective, injective, and flat modules. Tensor products. Modules over a PID and the Fundamental Structure Theorem. Applications to matrices (Rational Canonical Form, Jordan Canonical Form) and abelian groups (Classification Theorem).

A11: ALGEBRA III

- **I. Field Extensions:** Definitions. Degree of extension. Algebraic and transcendental extensions.
- **II. Algebraic Extensions:** Finite extensions. Minimal polynomial. Simple extensions. Finitely generated extensions. Transitivity of algebraic extensions. Splitting fields. Algebraic closure. Lifting of embeddings. Normal extensions. Separable extensions. Perfect fields. Primitive Element Theorem. Extensions with finite fields. Separable degree. Purely inseparable extensions. Separable and normal closure.
- **III. Galois Theory:** Galois group of an extension. Polynomials and Galois groups. Fixed fields. Galois extensions. Fundamental Theorem of Galois Theory. Radical extensions. Solvability of polynomial equations by radicals. Constructible numbers. Finite fields and Galois extensions. Roots of unity and cyclotomic extensions. Cyclic extensions. Norm and Trace. Discriminant. Kummer extensions. Hilbert's Theorem 90.

- **IV. Transcendental Extensions:** Transcendental elements. Transcendence bases. Noether Normalization Lemma. Lüroth's Theorem.

A12: ALGEBRAIC NUMBER THEORY

- **I.** Quadratic number fields.
- **II. Integral Dependence and Dedekind Domains:** Noetherian and Dedekind domains. Integral dependence. Ideal arithmetic and the final theorem.
- **III. Norm, Trace, Basis, and Discriminant:** Norm and Trace. Discriminant of an n-tuple. Free abelian groups of finite rank. Discriminant and integral basis of a field.
- **IV. Ideal Norm and Class Number:** Norm of ideals in algebraic number fields. Finiteness of the class number.
- **V. Reciprocity Laws:** Applications to quadratic and cyclotomic fields. Hilbert's ramification theory. Reciprocity laws. The discriminant theorem.
- **VI. Dirichlet's Unit Theorem:** Discrete subgroups of . Canonical embedding of an algebraic number field. Applications to the discriminant.

A13: GROUP REPRESENTATIONS Basic definitions and examples. Equivalent and irreducible representations. Modules and representations, indecomposable modules. Schur's Lemma, Wedderburn's Theorem, Maschke's Theorem. The regular representation and its decomposition. Character theory: definitions and examples. Orthogonality relations, number of characters. Character values, algebraic integers, and real characters. Brauer's Theorem. Character tables and information retrieval for groups. Applications: Abelian groups, groups of order , and Burnside's Theorem. Induced characters, Frobenius reciprocity. Normal subgroups and induced characters, Clifford's Theorems. Character extensions, Gallagher's Theorem.

A14: ALGEBRAIC GEOMETRY Affine varieties. Hilbert's Basis Theorem and Nullstellensatz. Polynomial functions, rational functions, and coordinate rings. The Zariski topology. The sheaf of regular functions on an affine variety. Functions and morphisms. Algebraic varieties. Projective space and projective varieties. Dimension. Rational and birational maps. Blow-ups. Smooth points and singularities. Smooth varieties. Hilbert polynomial of a projective variety. Degree of a projective variety. Intersection theory, Bézout's theorem, and applications. Selected topics: Schemes, Cohomology of Sheaves, Algebraic curves.

A20: SET THEORY Intuitive concept of a "set," Zermelo-Fraenkel axioms, power sets, construction of natural numbers, ordinal numbers and their arithmetic, transfinite induction, cardinal numbers, Axiom of Choice, Continuum Hypothesis, "Large Cardinals" and applications, elements of Descriptive Set Theory, special topics.

A21: LOGIC Propositional Calculus, truth tables, logical consequence, tautologies, formal proofs, Completeness and Compactness Theorems for Propositional Logic. Predicate Calculus, first-order languages, interpretations (models), satisfaction, Completeness and Compactness Theorems in Predicate Logic. Recursive functions, Gödel's Incompleteness Theorem. Interpretations of Set Theory and Peano Arithmetic. Properties expressible in first-order languages, multi-valued logic, elements of "λ-calculus," special topics.

A22: COMPUTABILITY Finite Automata (Finite State Machines), computability and non-computability with Finite Automata, Turing Machines, Post Systems, Recursive Functions, equivalence of various computational models, Halting Problem and undecidability, algorithmic complexity and measures, complexity examples, computability in Number Theory, Algebra, and Geometry, special topics.

A23: ALGORITHMS AND COMPLEXITY The concept of "Problem" in Algorithm Theory, time and space complexity, databases, techniques for finding "good" algorithms, balancing, dynamic programming, sorting algorithms, lower bounds of complexity, average complexity, graph algorithms, algorithms in Algebra and Geometry, Nondeterministically Polynomial (NP) problems, NP-complete problems, SAT problem, Hamilton problem, Clique problem, linear system solving in integers, special topics.

A30: COMBINATORICS Combinatorial principles using elementary arithmetic operations, Generating Functions, "Pigeonhole Principle," Combinatorial principles in Set Theory, "Polya Enumeration" using Group Theory, elements of Analytic Combinatorics, "Permutation Patterns," elements of Ramsey Theory, Graph Theory, special topics.

A31: CRYPTOGRAPHY

- **I. Historical Cryptosystems:** Caesar cipher, monoalphabetic substitution, polyalphabetic substitution, one-time-pad.
- **II. Block Ciphers:** Feistel scheme, DES, MACs.
- **III. Basic Algebraic/Number-Theoretic Algorithms:** Euclidean algorithm, Chinese Remainder Theorem, computations in the group of integers modulo .
- **IV. Public Key Systems:** Diffie-Hellman protocol (computational and decision problems, discrete logarithm problem). Encryption systems (ElGamal, RSA, integer factorization, attacks, and security definitions). Digital signature systems (ElGamal, RSA, DSA, Schnorr, attacks). Additional applications (e.g., commitment schemes).
- **V. Special Topics:** Elliptic curve cryptography, knapsack cryptosystems, lattices and the LLL basis reduction algorithm, code-based cryptography.

A32: CODING THEORY

- **I. Finite Fields:** Basic concepts and properties.
- **II. Linear Codes:** Definitions, linear algebra over finite fields, encoding and syndrome decoding.
- **III. Bounds:** Singleton bound, MDS codes, Hamming bound, perfect codes, Hamming codes, Gilbert-Varshamov bound.
- **IV. Constructions:** Methods for constructing new codes from old ones, Reed-Muller codes.
- **V. Cyclic Codes:** General construction, BCH codes, classical Reed-Solomon codes.
- **VI. Algebraic-Geometric Codes:** Generalized Reed-Solomon codes, Goppa codes.

Section B: Analysis

B0: REAL ANALYSIS Measures and outer measures, Lebesgue measure and Borel measures, measurable functions, Lusin's theorem, integration, Lebesgue and Lebesgue-Stieltjes integrals, product measures and multiple integrals, Tonelli and Fubini theorems. Modes of convergence (pointwise, in mean, in measure, almost uniform), signed and complex measures, Hahn, Jordan, and Lebesgue decompositions (absolutely continuous and singular measures). Hardy-Littlewood maximal function, Lebesgue differentiation theorem, Radon-Nikodym derivative, spaces and their duality. Additional topics: functions of bounded variation, absolutely continuous functions, Hilbert space theory, Fourier series in , Borel measures on (locally) compact spaces, and the Riesz Representation Theorem.

B1: FUNCTIONAL ANALYSIS Normed spaces, Banach spaces, separability, Riesz Lemma and compactness of the closed ball, uniformly convex norms, distance from a point to a closed convex set. Inner product spaces, Hilbert spaces, orthogonal complement, projections, orthonormal sets and bases, Bessel's inequality, Riesz-Fischer theorem, Parseval's identity, Schmidt theorem, dual space. Riesz Theorem for Hilbert spaces, Hahn-Banach Theorem, second dual, reflexivity, Baire category theorem. Uniform Boundedness Principle, weak and weak-* convergence, Mazur's theorem, Helly's theorem, Alaoglu's theorem. Bounded linear operators, the algebra , dual operator. Open Mapping Theorem, Closed Graph Theorem, spectrum (point, continuous, residual), spectral radius, compact operators, spectral theorem for compact self-adjoint operators. Special topics: Frechet spaces, Schauder's theorem, fixed point theorems, von Neumann ergodic theorem, Hilbert-Schmidt operators, Krein-Milman theorem.

B2: COMPLEX ANALYSIS Topology of the complex plane, extended complex plane, limits and continuity, series of numbers and functions (Weierstrass test), line integrals, derivatives and holomorphy, Cauchy-Riemann equations. Special functions (exponential, branches of logarithms, powers, roots). Cauchy-Goursat theorem, Taylor and Laurent series, singularities (poles, essential), Morera's theorem. Cauchy estimates, Liouville's theorem, Fundamental Theorem of Algebra, Identity Principle, Maximum Modulus Principle, Open Mapping Principle, winding number, homotopy, chains and cycles, homology, Residue Theorem, meromorphic functions, Argument Principle, Rouché's theorem. Simply connected domains, Montel's theorem, Riemann Mapping Theorem, conformal automorphisms of the disk and upper half-plane. Additional topics: Runge's theorem, Jensen's inequality, entire functions (Weierstrass factorization), Mittag-Leffler theorem, harmonic functions (Poisson formula, Dirichlet problem), Gamma and Zeta functions, Prime Number Theorem, analytic continuation.

B3: HARMONIC ANALYSIS Fourier series in and , Riemann-Lebesgue lemma, convergence criteria (Dini, Jordan), Dirichlet, Fejer, and Poisson kernels, approximations of identity, density of trigonometric polynomials. Fourier transform in , the Schwartz space and tempered distributions, Fourier transform in , Hausdorff-Young inequality. Translation-invariant operators, inversion formulas, Gauss and Poisson kernels, Marcinkiewicz theorem, Hardy-Littlewood maximal operator, Calderon-Zygmund decomposition, Hilbert transform in , multipliers, Bochner's theorem. Additional topics: BMO space, John-Nirenberg theorem, singular integrals, Riesz transforms.

B4: ERGODIC THEORY Examples of measurable dynamical systems, Poincaré Recurrence Theorem, Von Neumann and Birkhoff Ergodic Theorems, application to Ramsey theory (Furstenberg-Sarkozy theorem), equidistribution of sequences, normal numbers, continued fractions, Strong Law of Large Numbers for stationary processes. Weak and strong mixing, isomorphism, factors, Kronecker factor, Halmos-Von Neumann discrete spectrum theorem, invariant measures on compact metric spaces, uniquely ergodic systems, Furstenberg's multiple ergodic theorem, Roth's theorem, entropy of partitions and dynamical systems, non-isomorphism of Bernoulli shifts, Shannon-McMillan-Breiman theorem.

Section C: Geometry and Topology

Γ0: RIEMANNIAN GEOMETRY

- **I. Differentiable Manifolds:** Manifolds and maps. Tangent space and tangent bundle. Submanifolds. Vector fields and Lie derivative. Flows.
- **II. Connections:** Linear connections. Geodesics and the exponential map.
- **III. Riemannian Manifolds:** Riemannian metrics, Levi-Civita connection, normal coordinates, geodesics in model spaces.
- **IV. Geometry and Distance:** Distance and topology. Completeness and the Hopf-Rinow theorem. Isometries and the Myers-Steenrod theorem.
- **V. Curvature:** Curvature tensor, sectional and Ricci curvature. Riemannian submersions and O'Neil formulas. Bianchi identities, Schur's theorem, and Einstein manifolds.
- **VI. Geometry and Topology:** Jacobi equation. Conjugate points and Cartan-Hadamard-Kobayashi theorem. Spaces of constant curvature. Variation of length and Synge's formula. Bonnet-Myers theorem.

Γ1: DIFFERENTIABLE MANIFOLDS Differentiable manifolds and smooth maps. Quotient manifolds. Tangent space and bundle. Submanifolds. Constant Rank Theorem. Partitions of unity. Vector fields. Elements of Lie Groups and Lie Algebras. Differential forms: 1-forms, k-forms, exterior derivative, Lie derivative, and interior product. Integration: Orientations, manifolds with boundary, Stokes' Theorem. De Rham Theory: De Rham cohomology, Mayer-Vietoris sequence, homotopy invariance, cohomology calculations.

Γ2: ALGEBRAIC TOPOLOGY – HOMOTOPY

- **I. Homotopy:** Homotopic maps, homotopy type, categories, functors, and algebraic invariants. Path-connected components.
- **II. Fundamental Group:** Construction, examples, and applications. Free groups and free products. Seifert-Van Kampen theorem and its applications.
- **III. Covering Spaces:** Definitions and examples. Lifting properties. Covering maps and the fundamental group. Universal covers. Classification of covering spaces via subgroups.
- **IV. Higher Homotopy Groups:** H-groups and loop spaces. Suspensions. Homotopy groups, exact sequences, fibrations. Homotopy groups of spheres.

Γ3: ALGEBRAIC TOPOLOGY – HOMOLOGY

- **I. Homology:** Homotopy and homotopy type. Singular homology groups of a topological space. Chain complexes and exact sequences. Eilenberg-Steenrod axioms. Homotopy and Excision axioms for singular homology. Mayer-Vietoris sequence. Hurewicz theorem.

- **II. Homology with Coefficients:** Tensor product, torsion product, Universal Coefficient Theorem.
- **III. Cohomology:** Hom and Ext groups. Cohomology of chain complexes. Singular cohomology.
- **IV. Products:** Cross product, Eilenberg-Zilber Theorem, Künneth formula. Cup product and applications.
- **V. Topological Manifolds and Duality:** Orientation of topological manifolds. Cap product. Poincaré-Lefschetz Duality. Applications.

Γ4: GEOMETRY OF DYNAMICAL SYSTEMS Differential equations and integration of vector fields. Flows and the nature of trajectories. Invariants and minimal sets. Linear dynamical systems. Circle rotations. Monoparametric subgroups of the n-torus. Gradient vector fields. Shift on sequence spaces. Invariant measures. Unique ergodicity. Weyl's equidistribution theorem. Solenoids. Conjugacy. Logistic map. Smale horseshoe. Chaos. Sharkovskii's theorem. Poincaré-Bendixson theory. Flows on the 2-torus. Rotation number. Denjoy's theorem. C1-diffeomorphisms of Denjoy-Herman.

Section D: Differential Equations and Numerical Analysis

Δ10: PARTIAL DIFFERENTIAL EQUATIONS

- **Laplace Equation:** Harmonic functions, Harnack inequality, Fundamental solution, Green's functions, Dirichlet kernel, Poisson equation, Maximum principle, Energy methods, Perron's method.
- **Heat Equation:** Fundamental solution, Cauchy problem, Non-homogeneous problem, Mean value property, Maximum principle, Regularity, Energy methods.
- **Wave Equation:** Kirchhoff and Poisson formulas, Non-homogeneous problem, Energy methods.
- **Elliptic Equations:** Maximum principle, A priori estimates, Schauder estimates, Dirichlet problem.
- **Parabolic Equations:** Maximum principle, A priori estimates.
- **Nonlinear First-Order Equations:** Method of characteristics.

Δ11: PARTIAL DIFFERENTIAL EQUATIONS – THEORY OF WEAK SOLUTIONS

- **Sobolev Spaces:** Weak derivatives, properties, approximation by smooth functions, Extension, Trace, Sobolev and Morrey inequalities, Compactness, Dual spaces.
- **Elliptic Equations:** Weak solutions, existence, energy methods, Fredholm alternative, interior and boundary regularity, eigenvalues, eigenfunctions.
- **Parabolic & Hyperbolic Equations:** Weak solutions, Galerkin method, energy estimates, existence, uniqueness, and regularity of weak solutions.

- **Hyperbolic Systems:** Weak solutions, viscosity method, existence and uniqueness.

Δ12: ORDINARY DIFFERENTIAL EQUATIONS AND DYNAMICAL SYSTEMS Local existence (Picard-Lindelöf and Peano). Uniqueness. Gronwall's Lemma. Smooth dependence on data/parameters. Linear systems: fundamental solutions, constant/non-constant coefficients, asymptotic behavior. Asymptotic behavior of nonlinear equations. Stability and instability. Linearization. Lyapunov functionals. Poincaré-Bendixson, periodic solutions. Elements of bifurcation theory. Phase diagrams for autonomous systems.

Δ14: CALCULUS OF VARIATIONS Direct methods, existence of minimizers, coercivity, lower semi-continuity, weak solutions of Euler-Lagrange equations, convexity, systems, quasi-convexity, local minimizers, constraints, compensated compactness, concentration compactness, Palais-Smale condition, invariants, Noether's theorem, Pohozaev/Brezis-Nirenberg/Lions/Struwe results, effect of topology, isoperimetric inequalities.

Δ15: MATHEMATICAL FLUID THEORY I) Navier-Stokes equations for incompressible fluids. II) Functional spaces, embeddings. Riesz and Leray-Schauder theorems. III) Nonlinear stationary case: weak form, existence, and uniqueness. IV) Nonlinear non-stationary case: weak form, global and local solutions, Galerkin method. Existence/uniqueness for n=2 and n=3. V) Brief reference to compressible Navier-Stokes, Euler, and Prandtl equations.

Δ20: NUMERICAL ANALYSIS Norms and inner products, Cauchy-Schwarz, vector and matrix norms (Euclidean, max, Frobenius, p-norms). Holder and Minkowski inequalities. Equivalence of norms. Best approximations in inner product spaces. Condition number of a matrix. Perturbation analysis for linear systems. Solving systems with symmetric positive definite matrices: Cholesky decomposition, steepest descent, conjugate gradient methods. Banach Fixed Point Theorem. Newton's method for systems. IVP for ODEs: Euler, implicit Euler, Runge-Kutta, multi-step methods, A-stability. Interpolation and numerical integration.

Δ21: NUMERICAL SOLUTION OF DIFFERENTIAL EQUATIONS I) Hilbert spaces, Riesz representation, Lax-Milgram, Galerkin theorem. Sobolev spaces. II) Finite element spaces, piecewise polynomials, Bramble-Hilbert lemma, error estimates. III) Weak formulation of boundary value problems (1D and 2D elliptic PDEs). Finite element construction and error estimates. IV) Finite elements for heat and wave equations. V) Finite difference methods: stability, consistency, and convergence for 1D heat and Poisson equations.

Δ23: NUMERICAL LINEAR ALGEBRA I) Gaussian elimination (pivoting), LU and Cholesky decomposition. Sparse systems, backward error analysis. II) Least squares, QR decomposition (Householder/Givens), Singular Value Decomposition (SVD). III) Iterative methods: Jacobi, Gauss-Seidel, SOR, SSOR, Chebyshev. Krylov subspace methods (Arnoldi, GMRES, QMR, MINRES). IV) Preconditioning techniques. V) Approximation of eigenvalues and eigenvectors.

Section E: Probability and Statistics

E10: PROBABILITY THEORY Construction of probability spaces, random variables, stochastic processes, independence, expectation. Probabilistic method in combinatorics and number theory. Modes of convergence (almost sure, in mean square, in probability, in distribution). Kolmogorov's 0-1 law, Laws of Large Numbers, Three-series theorem, Khintchine's law of the iterated logarithm. Characteristic functions, Central Limit Theorem (Lindeberg condition). Conditional expectation, martingales, limit theorems, and applications.

E11: STOCHASTIC ANALYSIS Continuous stochastic processes. Brownian motion. Stopping times. Continuous martingales and basic properties. Doob-Meyer decomposition. Construction of Brownian motion. Itô integral for continuous martingales. Change of variables in stochastic integrals. Itô's formula and applications. Representation of continuous martingales via Brownian motion. Probabilistic study of Laplace and heat equations. Stochastic Differential Equations (SDEs): existence, uniqueness, examples, and solving special forms.