# Single vortices and vortex pairs in Bose-Einstein condensates

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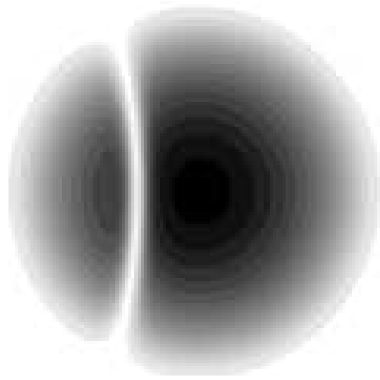
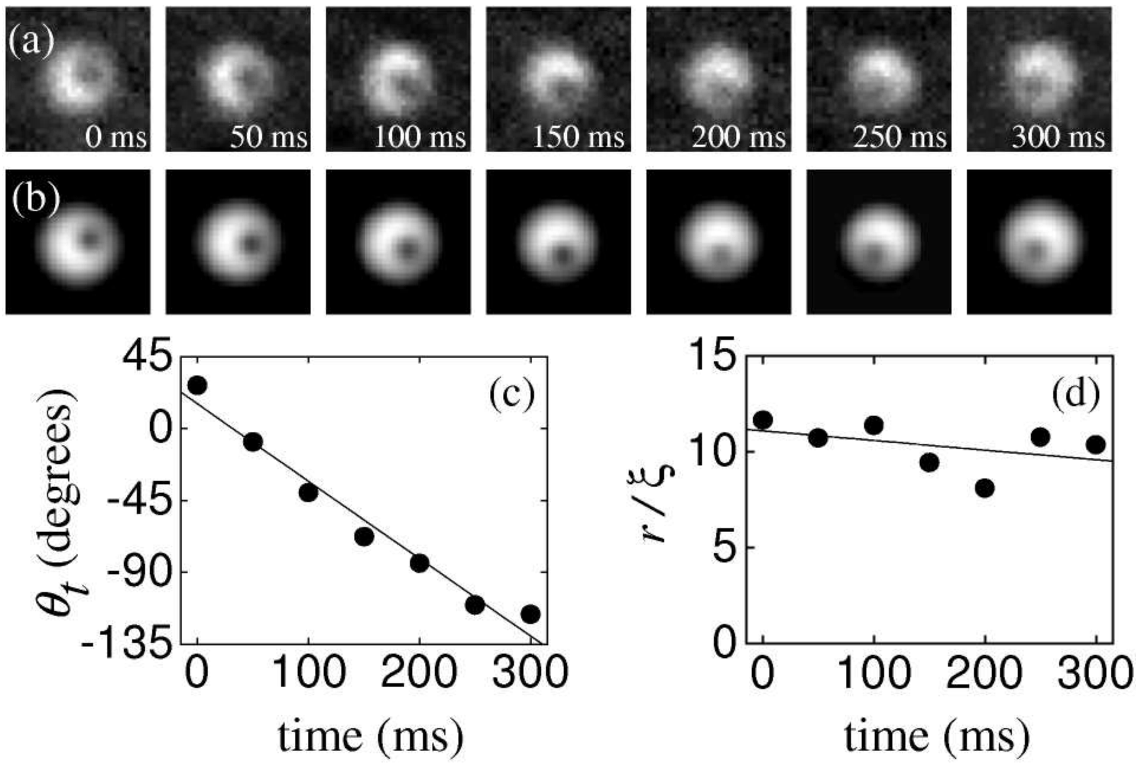
Phys. Rev. A **72**, 053609 (2005)

Phys. Rev. A **72**, 053624 (2005)

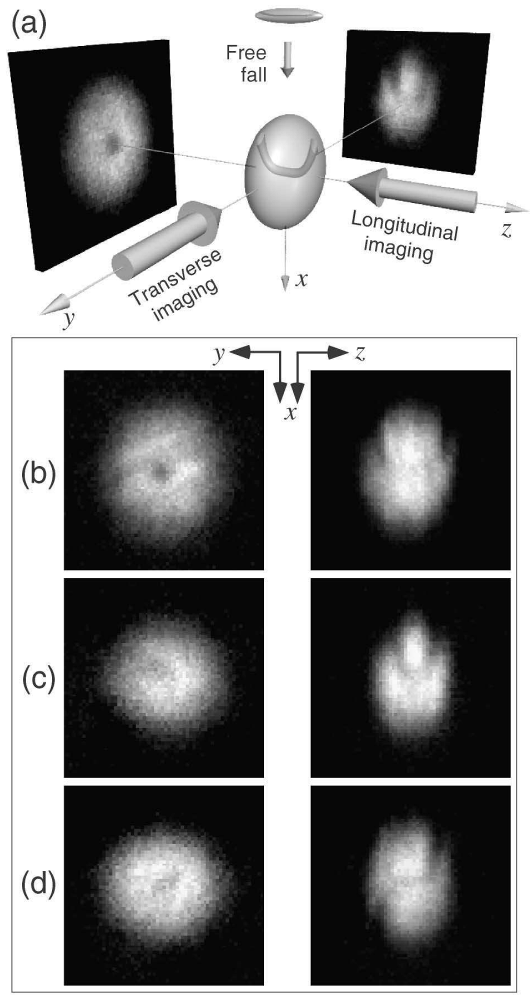


# Observation of precessing vortices




[Anderson, Haljan, Weiman, Cornell, PRL, 2000]

# Vortex lines




[Rosenbusch, Bretin, Dalibard, PRL, 2002] $\longrightarrow\longrightarrow$

# Formulation

We suppose the condensate wave function $\Psi(\boldsymbol{r}, t)$ and its energy functional [E.P. Gross, 1961; L.P. Pitaevskii, 1961]:

$$E = \frac{\hbar^2}{2M} \int |\boldsymbol{\nabla}\Psi|^2 \, d^3x + \int V \, |\Psi|^2 \, d^3x + 2\pi \, \frac{\hbar^2 a_s}{M} \int |\Psi|^4 \, d^3x.$$

Introduce the axisymmetric harmonic potential: $V(\rho, z) = \frac{1}{2} M(\omega_\perp^2 \, \rho^2 + \omega_\parallel^2 \, z^2)$, and appropriate units, to obtain the form:

$$E = \frac{1}{2} \int |\boldsymbol{\nabla}\Psi|^2 \, d^3x + \int V \, |\Psi|^2 \, d^3x + 2\pi \, \frac{N a_s}{a_\parallel} \int |\Psi|^4 \, d^3x, \quad V(\rho, z) = \frac{1}{2} (\rho^2 + \beta^2 z^2).$$

$a_{\perp,\parallel} \equiv \sqrt{\hbar/(M\omega_{\perp,\parallel})}$: units of length, $\beta \equiv \omega_\perp/\omega_\parallel$,

$N$: the number of atoms, $a_s$: the scattering length.

# Numerics for a vortex in a 3D trap

We introduce the chemical potential $\mu$ and substitute $\Psi(\boldsymbol{r},t) \to \Psi(\boldsymbol{r},t)\, e^{-i\mu t}$.

Thus: $E \to E - \mu N$.

A vortex precessing with angular frequency $\omega$ is an extremum of

$$E_{\rm rot} = E - \mu N - \omega \ell.$$

$$\ell = \frac{1}{i}\int \Psi^* \frac{\partial \Psi}{\partial \phi}\, d^3x \qquad \text{:angular momentum.}$$

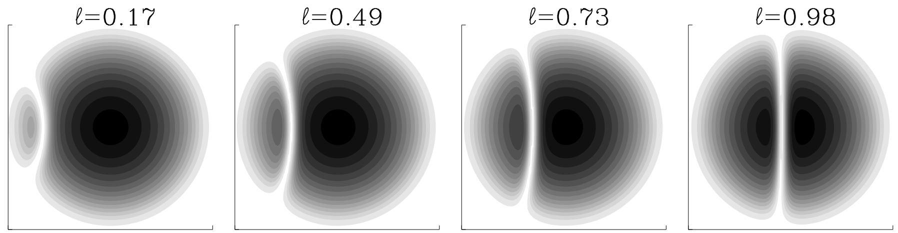
But: we use a norm-preserving relaxation algorithm to find extrema of the extended energy functional:

$$E_{\rm rot} = E - \mu N + \frac{a}{2}\,(\ell - b)^2, \qquad a, b : \text{const.}$$

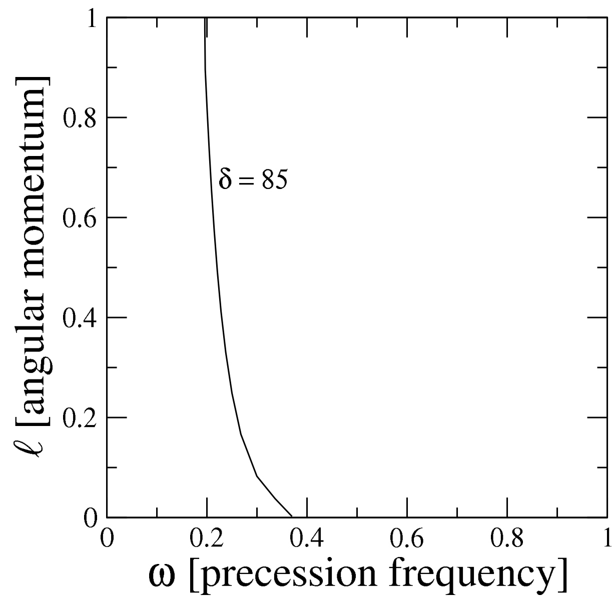
The solutions are precessing at an angular frequency $\omega = -a(\ell - b)$.

# A spherical trap

Experiment by [B.P. Anderson, *et al*, PRL **85**, 2857 (2000)].




The particle density $|\Psi|^2$ for vortex solutions corresponding to experiment.

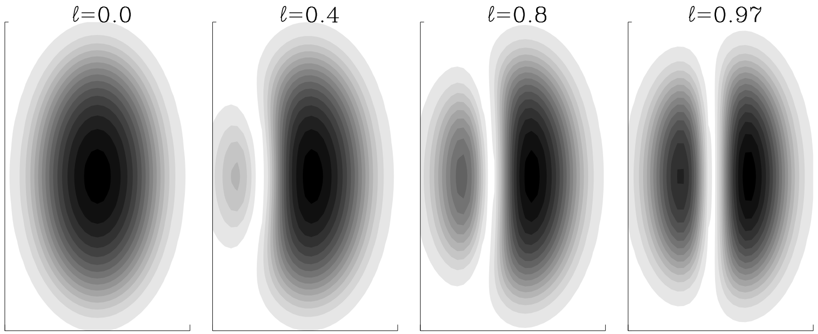



D.L. Feder *et al*, PRL **86**, 564 (2001) would give $\omega(\ell \to 0) = 2\pi \times 1.58$ Hz.

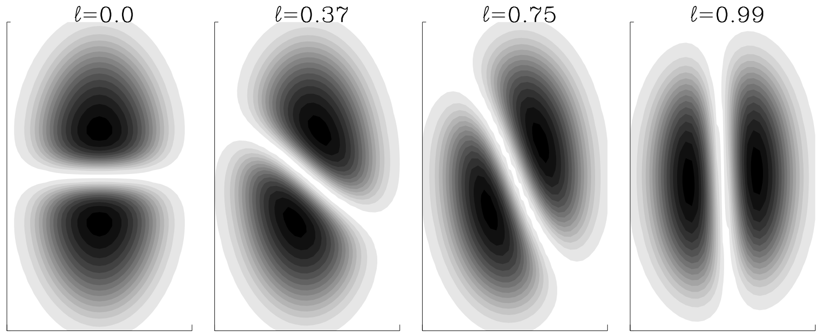
Angular momentum per particle $\ell$ (in units of $\hbar$) versus precession frequency $\omega$ (in units of trap frequency).

We have $1.52\text{Hz} < \omega/(2\pi) < 1.95\text{Hz}$ (e.g., 1.72 Hz for $\ell = 0.5$).

# An elongated trap




U-vortex.

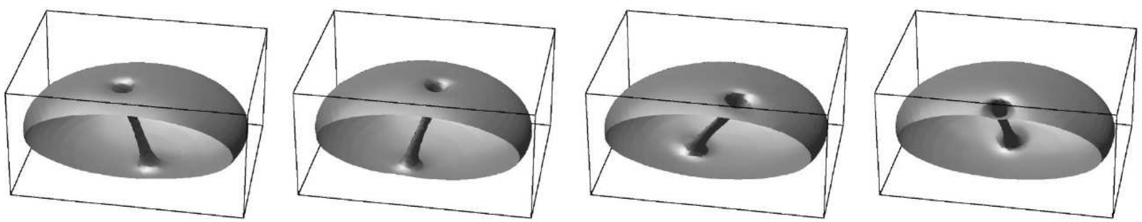



S-vortex.

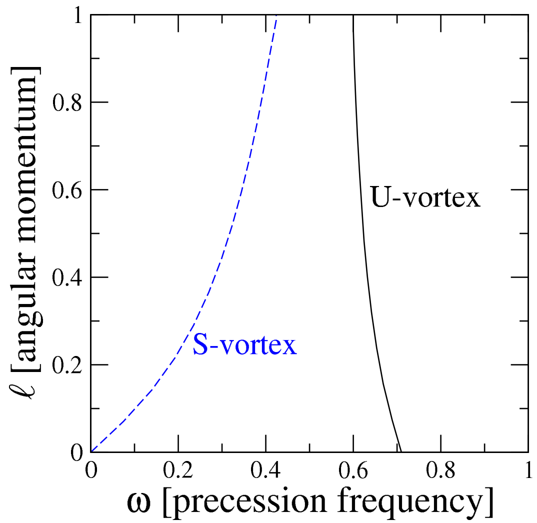
J.J. Garcia-Ripoll and V.M. Perez-Garcia, PRA **64** 53611 (2001).

M. Modugno, L. Pricoupenko, Y. Castin, EPJ D **22**, 235 (2003).

A. Aftalion and I. Danaila, PRA **68** 23603 (2003).



Smith, Heathcote, Krueger, Foot, PRL (2004)



# Lowest Landau level

The single (off-center) U-vortex mode in the LLL derived by O.K. Vorov *et al*, Phys. Rev. Lett. (2005):

$$\Psi = \frac{\ell^{\frac{1}{4}}}{\sqrt{\pi}}\,[(x-b)+iy]\,e^{-\frac{1}{2}[(x-a)^2-2iay+y^2]}$$

$$a = (\sqrt{\ell}-\ell)^{1/2}, \quad b = \frac{1-\ell}{a},$$

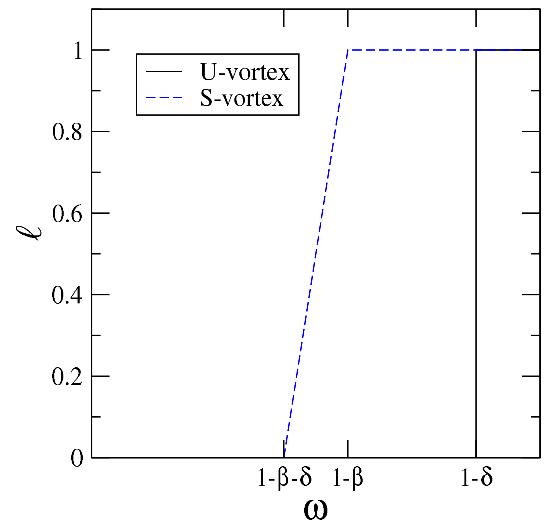
$0 \le \ell \le 1$, :angular momentum,

had been found numerically by [Butts and Rokshar, Nature (1998)].

# The S-vortex

corresponds to the linear combination:

$$\Psi_1 \sim z\, e^{-(\rho^2+\beta z^2)/2},$$
$$\Psi_2 \sim \rho\, e^{i\phi}\, e^{-(\rho^2+\beta z^2)/2},$$
$$\Psi = c_1\,\Psi_1 + c_2\,\Psi_2.$$




$\delta$: interaction strength

$\beta = \omega_\perp/\omega_\parallel$: trap aspect ratio

# Formulation II

We suppose the condensate wave function $\Psi(\boldsymbol{r}, t)$ and its equation of motion: [E.P. Gross, 1961; L.P. Pitaevskii, 1961]

$$i\,\frac{\partial \Psi}{\partial t} = -\frac{1}{2}\,\Delta\Psi + V(\rho)\,\Psi + \frac{g}{2}\,(\Psi^*\Psi)\Psi - \mu\Psi.$$

We have introduced the chemical potential $\mu$ and have substituted $\Psi(\boldsymbol{r}, t) \to \Psi(\boldsymbol{r}, t)\, e^{-i\mu t}$. $\mu$ should be chosen so that $\int (\Psi^*\Psi)\, d^3x = 1$.

$g$: the interaction strength.

[$1/\omega_\perp$: unit of time]

# Steady state precession: Equations of motion

Suppose a vortex in an axisymmetric trap, as a steady state precessing with frequency $\omega$. Its wavefunction is:

$$\bar{\Psi}(x, y, t) = \Psi(x', y'|\mu, \omega)\, e^{-i\mu t},$$

$$x' = x\cos\omega t + y\sin\omega t, \qquad y' = -x\sin\omega t + y\cos\omega t,$$

This satisfies the Gross-Pitaevskii equation ($\varepsilon_{\alpha\beta}$: antisymmetric tensor):

$$\mu\Psi - i\,\omega\,\varepsilon_{\alpha\beta}x_\alpha\partial_\beta\Psi = -\frac{1}{2}\Delta\Psi + V(\rho)\Psi + g\,(\Psi^*\Psi)\Psi. \tag{1}$$

where $V(\rho) = \frac{1}{2}\rho^2$ is the trapping potential, $g$ the interaction strength.

# Virial relations

Suppose a vortex located at $\boldsymbol{R} = (R, 0)$, with linear momentum $\boldsymbol{P} = (0, P)$.

Apply field-theoretical methods to the equations of motion to derive:

$$P - \omega\, R = 0, \qquad (I)$$

$$\omega P - \int \frac{x}{\rho}\frac{dV}{d\rho}\, n\, dxdy = 0, \qquad (II)$$

where $(\alpha = 1, 2)$

$$P_\alpha \equiv \int J_\alpha\, dxdy, \quad \text{the linear momentum,}$$

$$R_\alpha \equiv \int x_\alpha n\, dxdy, \quad \text{the mean position of the configuration,}$$

$$n = \Psi^*\Psi \text{ (particle density)}, \quad J_\alpha = \frac{1}{2i}\left(\Psi^*\partial_\alpha\Psi - \Psi\partial_\alpha\Psi^*\right) \text{ (current)}.$$

## Harmonic trap: $V = \frac{1}{2}\rho^2$

Virial relation I: $P - \omega R = 0$

Virial relation II: $\omega P - R = 0$

$$\Rightarrow P = 0 = R \qquad (\text{for } \omega \neq 1).$$

The result is satisfied by: (i) the calculated U- and S-vortices, (ii) the LLL closed form solution (by Vorov *et al*).

## Anharmonic trap: $V = \frac{1}{2}\rho^2 (1 + \lambda \rho^2)$

Virial relation I: $P - \omega R = 0$

Virial relation II: $\omega P - R = 2\lambda Q, \quad Q \equiv \int x\rho^2 n \, dx dy.$

$$\Rightarrow P = -\frac{2\lambda \omega}{1-\omega^2} Q, \quad R = -\frac{2\lambda}{1-\omega^2} Q,$$

# Vortices in the homogeneous Bose gas

The model is the nonlinear Schrödinger equation:

$$i\,\frac{\partial\Psi}{\partial t} = -\frac{1}{2}\,\Delta\Psi + (\Psi^*\Psi - 1)\Psi.$$

The energy functional is:

$$E = \frac{1}{2}\int\left[(\boldsymbol{\nabla}\Psi^* \cdot \boldsymbol{\nabla}\Psi) + (\Psi^*\Psi - 1)^2\right]\,d^3x.$$

A vortex has the form: $\Psi = f(\rho)\,e^{\pm i\phi}$, and the same energy for both signs.

$+$ for a "vortex",

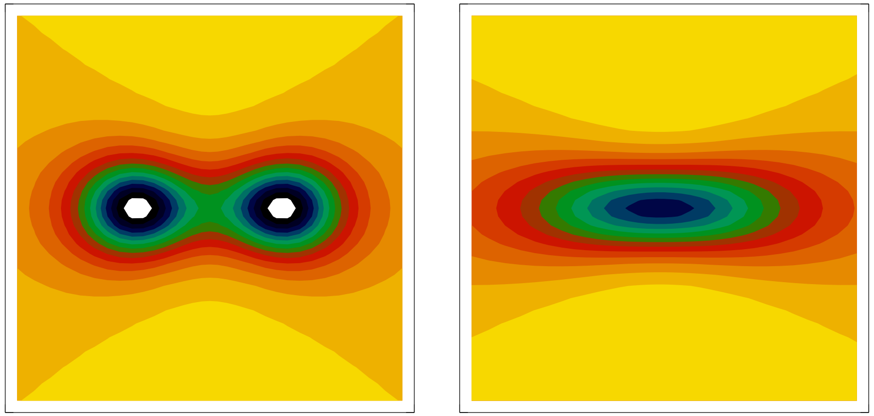
$-$ for a "antivortex".

# Vortex-Antivortex pairs – Solitary wave Droplets

[Jones and Roberts, J. Phys. A, 1982]

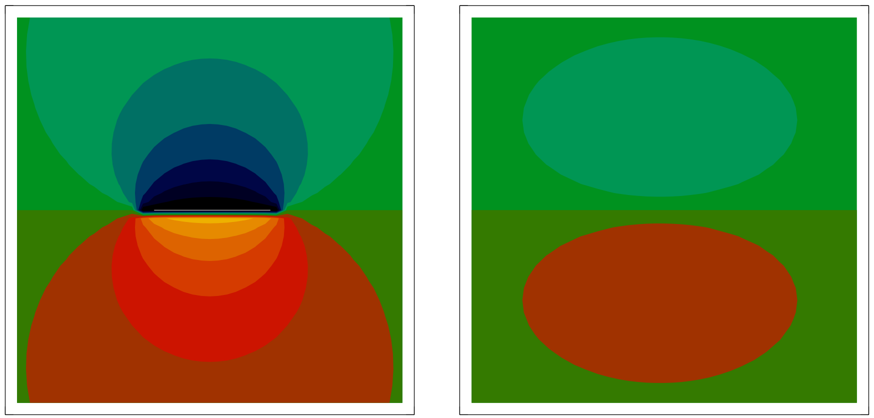
particle density

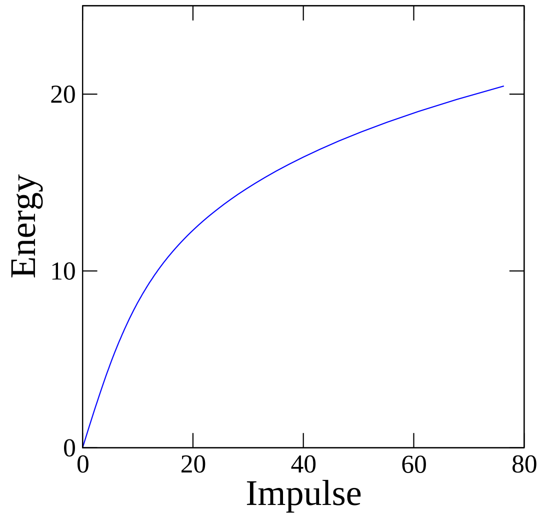
$\upsilon = 0.3$ $\upsilon = 0.8$



wavefunction phase

$\upsilon = 0.3$ $\upsilon = 0.8$





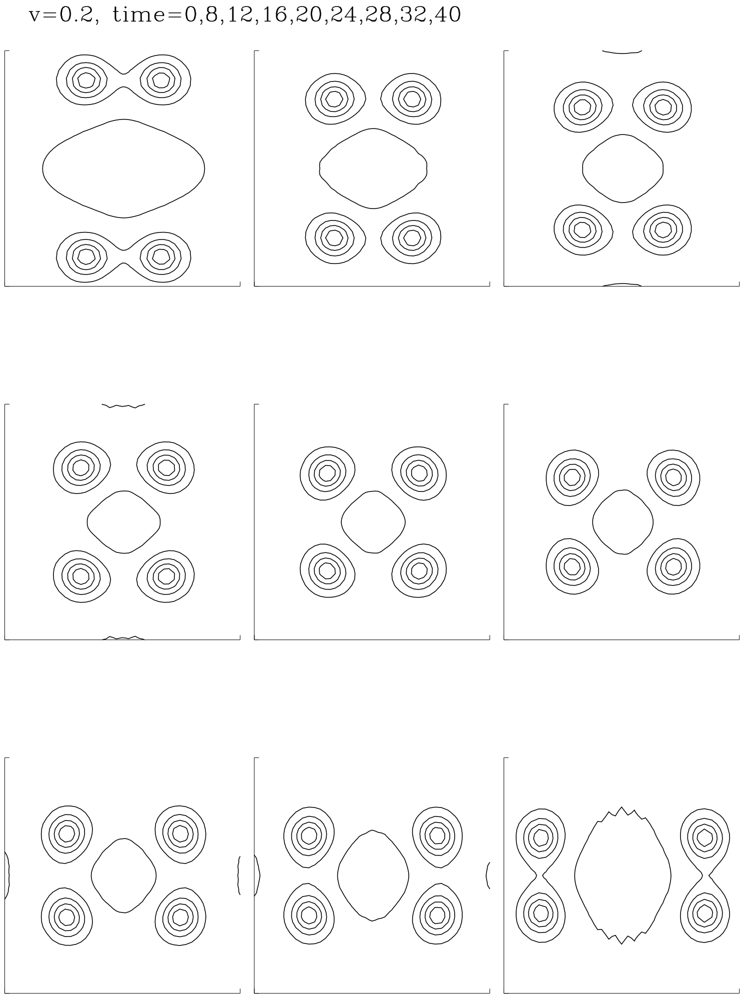

Impulse:

$$Q_\mu = \epsilon_{\mu\nu} \int x_\nu \, \gamma \, d^2x,$$

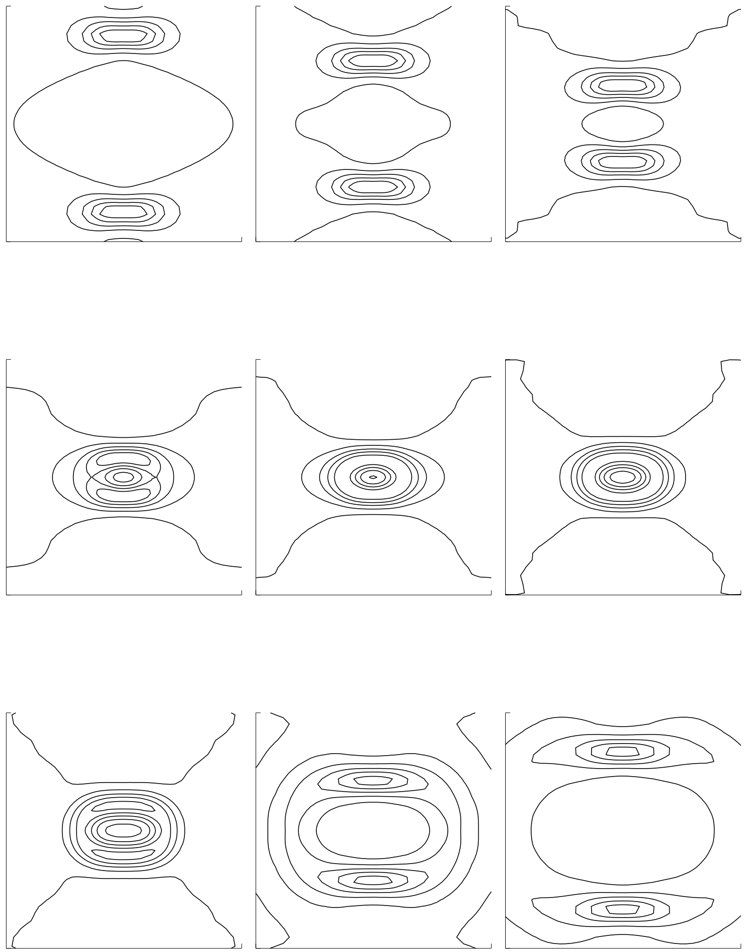
where $\gamma \equiv \frac{1}{i} \epsilon_{\mu\nu} \partial_\mu \Psi^* \partial_\nu \Psi$ can be called the local "vorticity", in correspondence to the hydrodynamic vorticity.

# Vortex-Antivortex pair collisions

$v = 0.2$



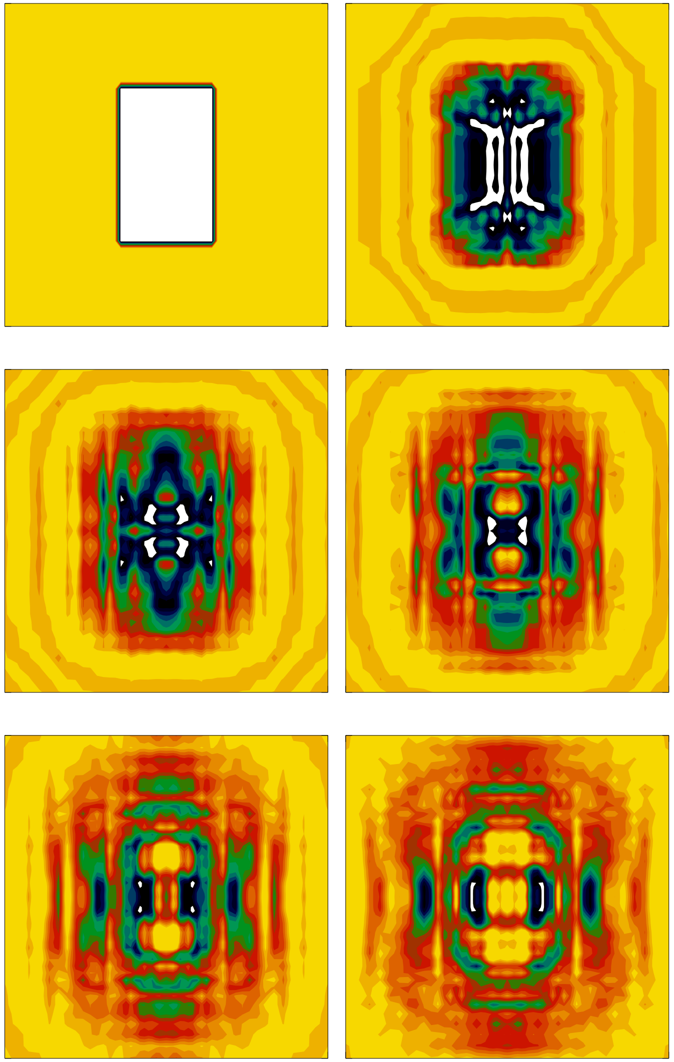

$v = 0.6$



# Vortex-antivortex pair generation

Deplete the superfluid (as, e.g., in N. Ginsberg, J. Brand, L.V. Hau, PRL, 2005)

particle density



wavefunction phase