Lecture 4. Continuous models

Literature

- Rajaraman, Solitons and Instantons, Sec. 2 (North Holland, 1982)
- Manton, Sutcliffe, Topological Solitons, Sec. 5 (CUP, 2004)

The Klein-Gordon equation (a linear equation)

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + m^2c^2\right)\phi = 0.$$

c is the velocity of traveling waves.

Example

(a) Find traveling wave solutions of the wave equation and of the Klein-Gordon equation. (b) Write the dispersion relation. (c) Write a linear combination of the above as a solution of the equation.

Example

Write the Lagrangian for the Klein-Gordon equation. Notice that this is in the following form, where the term $U(\phi)$ is called a potential,

$$\mathcal{L} = \frac{1}{2}\left[\frac{1}{c^2}(\partial_t\phi)^2 - (\partial_x\phi)^2\right] - U(\phi)$$

A nonlinear equation

A nonlinear model from the wave equation

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right)\phi + m^2 c^2 \phi^3 = 0.$$

Lagrangian and Energy

$$\mathcal{L} = \frac{1}{2}\left[\frac{1}{c^2}(\partial_t\phi)^2 - (\partial_x\phi)^2\right] - \frac{m^2c^2}{4}\phi^4.$$

$$E = \int \frac{1}{2}\left\{\left[\frac{1}{c^2}(\partial_t\phi)^2 + (\partial_x\phi)^2\right] + \frac{m^2c^2}{4}\phi^4\right\} dx.$$

- The energy is minimized for $\phi(x,t) = 0$ (note that it is positive definite).
- For localized solutions, an argument can be developed indicating that $\phi(x,t) \to 0$ as $x \to \infty$.

Lecture 4b. A nonlinear wave equation

The ϕ^4 model

We choose a more complicated potential

$$U(\phi) = \lambda(1-\phi^2)^2$$

giving a Lagrangian

$$L = \int \left[\frac{1}{2}(\partial_t \phi)^2 - \frac{1}{2}(\partial_x \phi)^2 - \lambda(1-\phi^2)^2 \right] dx$$

Note that the additional term in the potential is essentially a ϕ^4 term.

The Euler equation is

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} \right) \phi - 4\lambda(1-\phi^2)\phi = 0.$$

Simple wave solutions do not seem possible.

Static solutions of the equation

Time-independent solutions are $\phi = \phi(x)$. Euler's equation

$$\phi'' + 4\lambda(1-\phi^2)\phi = 0$$

Trivial static solutions of the ϕ^4 model

Trivial solutions are $\phi = \mathrm{const.}$ and we can see that $\phi = \pm 1$ are solutions of the equation.

Multiplying the Euler eqn by ϕ' makes it a total derivative

$$\frac{d}{dx}\left[\frac{1}{2}(\phi')^2 - \lambda(1-\phi^2)^2\right] = 0 \Rightarrow \frac{1}{2}(\phi')^2 - \lambda(1-\phi^2)^2 = 0$$

Note: we have set the integration constant to zero (why?).

A localized solution (kink)

$$\phi(x) = \pm\tanh\left(\sqrt{2\lambda}(x-\alpha)\right), \quad \alpha : \text{constant.}$$

More configurations

A kink at position α is

$$\phi(x) = \tanh\left(\sqrt{2\lambda}(x-\alpha)\right)$$

An antikink at position α is

$$\phi(x) = -\tanh\left(\sqrt{2\lambda}(x-\alpha)\right)$$

Kink-antikink

We can write a configuration which represents a kink at position α and an antikink at position $-\alpha$,

$$\phi(x) = \tanh\left(\sqrt{2\lambda}(x-\alpha)\right) - \tanh\left(\sqrt{2\lambda}(x+\alpha)\right) + 1.$$

This is valid as long as α is large.
But, this in not a solution of the static equation.

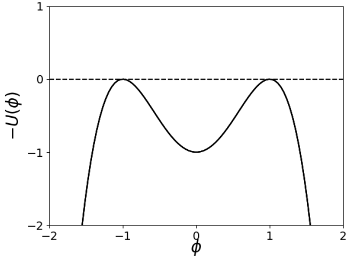
Localized solutions

We may give a picture of the above method of integration in terms of the motion of a Newtonian system for an effective energy function.

$$\frac{1}{2}(\phi')^2 - \lambda(1-\phi^2)^2 = 0 \quad \text{or} \quad \frac{1}{2}(\phi')^2 - U(\phi) = 0.$$

Consider x as time and ϕ as position of a particle. The above is formally the same as the law of energy conservation.

Note: the expression on the left is not the energy of a real particle.



Traveling solitary waves

A traveling wave

The following form satisfies the equation

$$\phi(x,t) = \tanh\left(\sqrt{2\lambda}\gamma(x - vt)\right), \quad \gamma = \frac{1}{\sqrt{1-v^2}}.$$

The above is obtained from the static solution by the transformation

$$x \to \frac{x - vt}{\sqrt{1-v^2}}$$

called the Lorentz transformation.

The Sine-Gordon equation (only time-independent)

Lagrangian

$$\mathcal{L} = \frac{1}{2}(\phi')^2 + (1 - \cos\phi)$$

Euler's equation

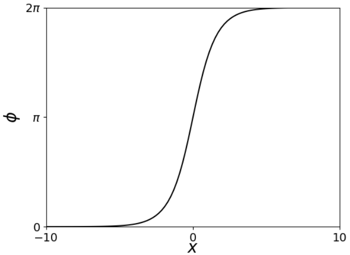
$$\phi'' - \sin\phi = 0.$$

A solitary wave solution is

$$\phi(x) = 4\tan^{-1}(e^{x-\alpha}).$$

We have

$\phi(x \to -\infty) = 0$
$\phi(x \to \infty) = 2\pi.$



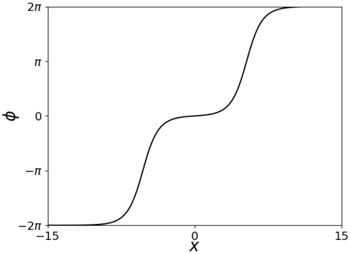
Multi-kinks

Example

Construct the picture of a particle moving in a potential. Note that this has extrema at $\phi = 0, \pi, 2\pi, \ldots$..

We may construct multi-kink configurations for the sine-Gordon equation, such as

$$\phi(x) = 4\tan^{-1}\left(\frac{\sinh(x)}{\alpha}\right), \qquad \alpha : \text{const.}$$



Lagrangian for two fields

Lagrangian for two fields

Consider two fields $\phi_1(x), \phi_2(x)$ and the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial_x\phi_i\,\partial_x\phi_i - \frac{m^2}{2}\phi_i\phi_i, \qquad i = 1, 2$$

It is understood that we sum over i.

Example

Derive the Euler-Lagrange equations.

Complex fields

Lagrangian for a complex field

We define the complex field

$$\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2), \quad \phi^* = \frac{1}{\sqrt{2}}(\phi_1 - i\phi_2)$$

the above Lagrangian density is written as

$$\mathcal{L} = \partial_x \phi^* \, \partial_x \phi - m^2 \phi^* \phi.$$

Example

Derive the Euler-Lagrange equations for $\phi \in \mathbb{C}$.

$$\frac{\delta L}{\delta \phi^*} = 0 \Rightarrow \partial_x^2 \phi + m^2 \phi = 0.$$

Complex fields in many space dimensions

Lagrangian for a complex field in many dimensions

$$\mathcal{L} = \partial_\mu \phi^* \, \partial_\mu \phi - m^2 \phi^* \phi.$$

where $\mu = 1, 2, \ldots$ is summed over its values.

Example

Write explicitly the Lagrangian in two space dimensions.

Example

Derive the Euler-Lagrange equations.