Magnetic materials:
domain walls, vortices, and bubbles
(lecture notes)

Stavros Komineas

Department of Applied Mathematics, University of Crete

ABSTRACT. We introduce the notion of the magnetization vector and present the Landau-Lifshitz equation which governs its static and dynamical properties. Subsequently, we focus on topological magnetic structures such as domain walls, vortices, and bubbles. We give a description of such structures based on the notions of the topological numbers which are relevant for the magnetization vector. Finally, the dynamics of vortices and bubbles is studied based on a link between topology and dynamics.

The lectures are accompanied with analytical and numerical exercises on the Landau-Lifshitz equation and some of its interesting solutions.

[Date for present version (Cambridge, 10/5/2012)] .

Contents

CHAPTER 1

The magnetization vector and the Landau-Lifshitz equation

1. Magnetic moment of atoms

1.1. Magnetic moments. In classical electromagnetism we may consider a current in a closed loop [**1, 2**]. The *magnetic moment* is defined as an integral over the loop

$$\boldsymbol{\mu} = \frac{I}{2} \oint_C \boldsymbol{r} \times d\boldsymbol{s}, \tag{1.1}$$

where I is the current [units A m^2]. Another useful form of the above integral is

$$\boldsymbol{\mu} = I \int_A d\boldsymbol{a} = I A\, \hat{\boldsymbol{n}},$$

where A denotes both the surface enclosed by the loop C and the area of this domain, and $\hat{\boldsymbol{n}}$ is the directed normal to the surface A.

Form (1.1) shows that the magnetic moment $\boldsymbol{\mu}$ associated with an orbiting electron is proportional to the angular momentum $\boldsymbol{L}$ of the electron motion. Thus we write

$$\boldsymbol{\mu} = \gamma \boldsymbol{L},$$

where $\gamma = g_e |e|/2m_e$ is the gyromagnetic ratio. The charge and mass of the electron are e, m_e, while g_e is the Landé factor which may take values different than unity depending on whether the magnetic moment is due to pure orbital motion or to the electron spin angular momentum.

1.2. Precession. The energy E of a magnetic moment in a magnetic field $\boldsymbol{B}$ can be written in the form [**1, 2**]

$$E = -\boldsymbol{\mu} \cdot \boldsymbol{B} \tag{1.2}$$

so the energy is minimized when the magnetic moment is aligned with the magnetic field. As the magnitude of the magnetic moment is constant (and we suppose a constant in time magnetic field), we may write $E = -\mu B \cos\psi$, where ψ is the angle between $\boldsymbol{\mu}$ and $\boldsymbol{B}$. We then see that we may define a torque with magnitude

$$\tau = -\frac{\partial E}{\partial \psi} = -\mu B \sin\psi,$$

which is equal to $-|\boldsymbol{\mu} \times \boldsymbol{B}|$. The direction of the torque should be perpendicular to both $\boldsymbol{\mu}$ (as its magnitude should remain constant) and to $\boldsymbol{B}$ as the energy, and thus the angle ψ, should be conserved. We can finally write

$$\boldsymbol{\tau} = \boldsymbol{\mu} \times \boldsymbol{B}. \tag{1.3}$$

[The sign of the above relation is chosen so as the force derived from the energy be $\boldsymbol{F} = -\boldsymbol{\nabla}(E)$.] We can now write an equation for the time derivative of the angular momentum or, more conveniently, of the magnetic moment

$$\frac{d\boldsymbol{\mu}}{dt} = \gamma\, \boldsymbol{\mu} \times \boldsymbol{B}. \tag{1.4}$$

Let us suppose a coordinate system where we set the axis z in the direction of the field $\boldsymbol{B} = B\hat{\boldsymbol{z}}$. We write $\boldsymbol{\mu} = (\mu_x, \mu_y, \mu_z)$ and then Eq. (1.4) reads

$$\begin{aligned} \dot{\mu}_x &= \gamma B \, \mu_y \\ \dot{\mu}_y &= -\gamma B \, \mu_x \\ \dot{\mu}_z &= 0. \end{aligned} \tag{1.5}$$

The solution to the latter system of equations is

$$\begin{aligned} \mu_x &= \mu \, \sin\theta \cos(\omega_L t) \\ \mu_y &= \mu \, \sin\theta \sin(\omega_L t) \\ \mu_z &= \mu \, \cos\theta, \end{aligned}$$

where θ is the angle between $\boldsymbol{\mu}$ and $\boldsymbol{B}$, and $\omega_L = \mu B$ is called the *Larmor* precession frequency. We thus see that μ_z remains constant while the component of the magnetic moment perpendicular to the external field is precessing around the vector $\boldsymbol{B}$.

Example 1.1.

We can recast the equation for a magnetic moment $\boldsymbol{\mu}$ in an external field $\boldsymbol{B}$ using a complex variable:

$$\mu_c = \mu_x + i\mu_y.$$

The first two of the equations of motion (1.5) are written in the form

$$\dot{\mu}_c = -i\,\omega_L\,\mu_c$$

of which the solution is written in the convenient form

$$\mu_c = \mu_0\, e^{-i\omega_L t}. \qquad \mu_0 : \text{constant.} \; \square$$

2. The magnetization vector

We suppose that all atoms in a specific material have a magnetic moment with the same magnitude, or that we can attribute to each lattice site in a solid material a magnetic moment with a certain constant magnitude. The magnetization properties of the material are defined by the atomic magnetic moments. In a *ferromagnetic* material the vector for the magnetic moment varies only slowly in space and it is then useful to treat the underlying material as a *ferromagnetic continuum.* That is, we may define the total magnetic moment in the unit volume of the material, or the density of magnetic moment $\boldsymbol{M}$. We write approximately

$$\boldsymbol{M} = \frac{\Delta\boldsymbol{\mu}}{\Delta V}$$

where $\Delta\boldsymbol{\mu}$ is the total magnetic moment in a volume element ΔV. The magnetization $\boldsymbol{M}$ has units [Ampere/meter] in SI.

The magnetic moment density $\boldsymbol{M}$ is called the *magnetization vector.* As it gives the local density of the magnetic moments it is a function of position and maybe of time: $\boldsymbol{M} = \boldsymbol{M}(\boldsymbol{r}, t)$. As the magnetic moments of atoms are constant in magnitude the magnetization vector $\boldsymbol{M}$ is also considered to have a length which is constant in time. This is expressed by:

$$\boldsymbol{M}^2 = M_s^2,$$

where M_s is called the saturation magnetization. The saturation magnetization can easily be measured when a magnetic sample is fully magnetized (saturated) along a certain direction (e.g., by use of a strong magnetic field).

3. Energy and equation of motion

3.1. Magnetic energy. A ferromagnetic material is characterized by the property that neighbouring magnetic moments tend to be aligned with each other. For a chain of magnetic spins (moments) $\boldsymbol{S}_i$ an interaction which favours alignment of spins is typically expressed by the *exchange interaction* of the form

$$-J\sum_i \boldsymbol{S}_i\cdot\boldsymbol{S}_{i+1} \tag{3.1}$$

where J is the exchange constant. While this is the form of the exchange interaction for a discrete system of magnetic spins, these notes will only be concerned with a description of the continuum. In order to derive a model for the continuum we first assume the magnetization vector $\boldsymbol{M}$ at discrete points α,β, etc, and treat them as classical vectors $\boldsymbol{M}_\alpha,\boldsymbol{M}_\beta$. Assuming that α,β are neighbouring sites we have the exchange contribution

$$-\frac{JS^2}{M_s^2}\,\boldsymbol{M}_\alpha\cdot\boldsymbol{M}_\beta.$$

For a continuum model to make sense we further assume that the magnetization vector varies slowly between neighbouring sites. Let us take α,β to lie on the x-axis, with a the lattice spacing, and write

$$\boldsymbol{M}_\beta\approx\boldsymbol{M}_\alpha+a\frac{\partial\boldsymbol{M}_\alpha}{\partial x}+\frac{a^2}{2}\frac{\partial^2\boldsymbol{M}_\alpha}{\partial x^2},$$

where the notation indicates that the derivatives are calculated at site α. For the neighbour of $\boldsymbol{M}_\alpha$ on the opposite site we have a similar relation where $a\to -a$, furthermore, similar relations hold for the neighbours in the y,z directions. Thus the contribution to the exchange energy from the area of $\boldsymbol{M}_\alpha$ is

$$-JS^2\left[6+\frac{a^2}{M_s^2}\,\boldsymbol{M}_\alpha\cdot\partial_\mu\partial_\mu\boldsymbol{M}_\alpha\right].$$

The index μ takes values $\mu=1,2,3$ and we employ the Einstein summation convention (the repeated index μ is summed over its three values). The first term in the above form of the energy can be dropped as it is only a constant. We finally integrate over all space to find

$$E_{\rm ex}=-\frac{A}{M_s^2}\int(\boldsymbol{M}\cdot\partial_\mu\partial_\mu\boldsymbol{M})\,d^3\boldsymbol{r}\Rightarrow$$

$$E_{\rm ex}=\frac{A}{M_s^2}\int(\partial_\mu\boldsymbol{M}\cdot\partial_\mu\boldsymbol{M})\,d^3\boldsymbol{r}. \tag{3.2}$$

We have introduced the *exchange constant* A which has units of [Joule/meter], and the integration is extended over the volume of the magnetic material. The second form of the exchange energy is obtained from the first by a partial integration (divergence theorem).

The main property of a ferromagnet which is implied by the exchange energy (3.2) is that the magnetization should tend to be uniform, or, $\partial_\mu\boldsymbol{M}=0$. On the other hand, the direction of the uniform magnetization is arbitrary, that is, the exchange energy term (3.2) is isotropic.

It is commonly seen that there are preferred directions in space for the orientation of magnetic moments, which depend on the crystal lattice of the material. We call this property the *magnetocrystaline anisotropy* or simply *magnetic anisotropy.* The simplest case is *uniaxial magnetic anisotropy* and it can be expressed by an energy term of the form

$$E_{\rm a}=\frac{K}{M_s^2}\int(M_3)^2\,d^3\boldsymbol{r}, \tag{3.3}$$

where K is the anisotropy constant (units [Joule/meter^3]). If we assume $K>0$, the energy term (3.3) disfavours the third component M_3 of the magnetization over the other two components (M_1,M_2). Such an energy term gives rise to *easy-plane* anisotropy, that is, the magnetization vector prefers to lie in the xy-plane. In the case $K < 0$ the M_3 component is favoured and we would thus call the above an *easy-axis* anisotropy term.

In order to discuss one more important interaction which exists in ferromagnets we need to understand that the aligned magnetic moments create a magnetic field which can be significant. This is called the *magnetostatic field* (also called the *demagnetizing field*) $\boldsymbol{H}_m$ and it is given by Maxwell's equations

$$\nabla \cdot \boldsymbol{H_m} = -\nabla \cdot \boldsymbol{M}, \qquad \nabla \times \boldsymbol{H_m} = 0. \tag{3.4}$$

The magnetostatic field interacts with the magnetization and gives rise to the *magnetostatic energy* term

$$E_{\mathrm{m}} = -\frac{1}{2}\mu_0 \int \boldsymbol{M} \cdot \boldsymbol{H}_m \, d^3\boldsymbol{r}. \tag{3.5}$$

If the magnet is placed in an *external magnetic field* $\boldsymbol{H}_{\mathrm{ext}}$ then this gives rise to the *Zeeman energy* term

$$E_{\mathrm{ext}} = -\mu_0 \int \boldsymbol{M} \cdot \boldsymbol{H}_{\mathrm{ext}} \, d^3\boldsymbol{r}. \tag{3.6}$$

We are now ready to write the total magnetic energy as the sum of exchange, anisotropy, magnetostatic, and external field energies:

$$E = \frac{A}{M_s^2} \int \partial_\mu \boldsymbol{M} \cdot \partial_\mu \boldsymbol{M} \, d^3\boldsymbol{r} + \frac{K}{M_s^2} \int (M_3)^2 \, d^3\boldsymbol{r} - \frac{1}{2}\mu_0 \int \boldsymbol{M} \cdot \boldsymbol{H}_m \, d^3\boldsymbol{r} - \mu_0 \int \boldsymbol{M} \cdot \boldsymbol{H}_{\mathrm{ext}} \, d^3\boldsymbol{r}. \tag{3.7}$$

3.2. Energy: rationalized units. It will be very convenient for further calculations to rationalize the expression for the energy (for an introduction to this method see, e.g., [**3**]). First, it is natural to normalize $\boldsymbol{M}$, as well as all quantities with the same units, to the saturation magnetization M_s. We define the rationalized fields:

$$\boldsymbol{m} \equiv \frac{\boldsymbol{M}}{M_s}, \qquad \boldsymbol{h}_m \equiv \frac{\boldsymbol{H}_m}{M_s}, \qquad \boldsymbol{h}_{\mathbf{ext}} \equiv \frac{\boldsymbol{H}_{\mathrm{ext}}}{M_s}. \tag{3.8}$$

We further notice that the exchange term in the energy contains space derivatives, and therefore the ratio of the constant multiplying the exchange integral to, e.g., the constants of the magnetostatic integral would produce a natural length scale for the system. This motivates the definition of the *exchange length*

$$\ell_{\mathrm{ex}} \equiv \sqrt{\frac{2A}{\mu_0 M_s^2}}. \tag{3.9}$$

Substituting the definitions (3.8) in the energy (3.7) and measuring length in units of exchange length (i.e., substitute $\boldsymbol{r} \to \ell_{\mathrm{ex}}\boldsymbol{r}$), we obtain

$$E = (\mu_0 M_s^2 \, \ell_{\mathrm{ex}}^3) \left[\frac{1}{2} \int \partial_\mu \boldsymbol{m} \cdot \partial_\mu \boldsymbol{m} \, d^3\boldsymbol{r} + \frac{q}{2} \int (m_3)^2 \, d^3\boldsymbol{r} - \frac{1}{2} \int \boldsymbol{m} \cdot \boldsymbol{h}_m \, d^3\boldsymbol{r} - \int \boldsymbol{m} \cdot \boldsymbol{h}_{\mathrm{ext}} \, d^3\boldsymbol{r} \right]. \tag{3.10}$$

We will assume in the following that the energy is measured in units of $(\mu_0 M_s^2 \ell_{\mathrm{ex}}^3)$ so that the energy of the system is simply given by the term in the square brackets. The only constant remaining in the definition of the energy is multiplying the anisotropy term, it is called the *quality factor*

$$q \equiv \frac{2K}{\mu_0 M_s^2} \tag{3.11}$$

and measures the strength of the anisotropy relative to the magnetostatic term.

Let us summarize here the rationalized form of the energy terms discussed in this section. The total energy is the sum of the exchange energy

$$E_{\rm ex} = \frac{1}{2} \int \partial_\mu \boldsymbol{m} \cdot \partial_\mu \boldsymbol{m}\, d^3\boldsymbol{r}, \tag{3.12}$$

the anisotropy energy, which may be easy-plane

$$E_{\rm a} = \frac{q}{2} \int (m_3)^2\, d^3\boldsymbol{r} \tag{3.13}$$

or easy-axis

$$E_{\rm a} = \frac{q}{2} \int [(m_1)^2 + (m_2)^2]\, d^3\boldsymbol{r}, \tag{3.14}$$

the magnetostatic energy

$$E_{\rm m} = -\frac{1}{2} \int \boldsymbol{m} \cdot \boldsymbol{h}_m\, d^3\boldsymbol{r}, \tag{3.15}$$

and the external field energy

$$E_{\rm ext} = -\int \boldsymbol{m} \cdot \boldsymbol{h}_{\rm ext}\, d^3\boldsymbol{r}. \tag{3.16}$$

Example 3.1.

We have for Permalloy $A = 1.3 \times 10^{-11}$ J/m and $M_s = 0.69 \times 10^6$ A/m. We find

$$\ell_{\rm ex} = 6.59\,\text{nm}, \qquad \mu_0 M_s^2 \ell_{\rm ex}^3 = 1.71 \times 10^{-19}\,\text{Joule}.\square$$

Example 3.2. *By comparing the exchange and the anisotropy term find a natural length scale for the system and give a rationalized form of the energy.*

The ratio A/K has units [length]^2, so we define the quantity

$$\delta \equiv \sqrt{\frac{A}{K}}, \tag{3.17}$$

and we note that $\delta = \ell_{\rm ex}/q$. Using δ as the unit of length (i.e, substituting $\boldsymbol{r} \to \delta \boldsymbol{r}$) in (3.7) we obtain

$$E = (\mu_0 M_s^2\, \delta^3 q) \left[\frac{1}{2} \int \partial_\mu \boldsymbol{m} \cdot \partial_\mu \boldsymbol{m}\, d^3\boldsymbol{r} + \frac{1}{2} \int (m_3)^2\, d^3\boldsymbol{r} - \frac{1}{2q} \int \boldsymbol{m} \cdot \boldsymbol{h}_{\rm m}\, d^3\boldsymbol{r} - \frac{1}{q} \int \boldsymbol{m} \cdot \boldsymbol{h}_{\rm ext}\, d^3\boldsymbol{r} \right]. \tag{3.18}$$

This form indicates that a large quality factor weakens the effects arising from the magnetostatic (and the external) field. □

Exercise 3.3. *Suppose a thin film which is infinite in the xy-plane and the axis z is perpendicular to the film. Show that for a thin film which is uniformly magnetized along the z-axis, Maxwell's equations give a magnetic field*

$$\boldsymbol{h}_{\rm m} = -(0, 0, m_z)$$

inside the film, and $\boldsymbol{h}_{\rm m} = 0$ *outside the film.* □

3.3. The Landau-Lifshitz equation. We assume that the system is hamiltonian with an energy (3.7). We then only need to write Hamilton's equations as the equations of motion for the magnetization $\boldsymbol{M}$. In order to do this we need the Poisson bracket relations between the dynamical variables which are the components of the magnetization [**5**]. We shall rather follow here a simpler approach. We note first that the time derivatives of the canonical variables in the Hamiltonian formalism are given by the variation of $\delta E/\delta \boldsymbol{M}$. This variation of the energy gives the fields acting on the magnetization. In the present problem, however, we should also consider the constraint that the length of $\boldsymbol{M}$ is constant. Inspired by the equation for the precession of a magnetic moment around an external magnetic field we may write an equation of motion for the magnetization which precesses around the field $\delta E/\delta \boldsymbol{M}$. This is called the Landau-Lifshitz (LL) equation [**7**]:

$$\frac{\partial \boldsymbol{M}}{\partial t} = -\gamma \boldsymbol{M} \times \boldsymbol{F}, \qquad \boldsymbol{F} \equiv -\frac{\delta E}{\delta \boldsymbol{M}}. \tag{3.19}$$

We can calculate the variation

$$-\frac{\delta E}{\delta \boldsymbol{M}} = \mu_0 \left[\frac{2A}{M_s^2} \Delta \boldsymbol{M} - \frac{2K}{M_s^2} M_3 \hat{\boldsymbol{e}}_3 + \boldsymbol{H}_m + \boldsymbol{H}_{\text{ext}} \right],$$

where $\hat{\boldsymbol{e}}_3$ is the unit vector along the third direction of the magnetization. The derivation of the last formula is easy except probably from the magnetostatic field term [**6**].

Eq. (3.19) has some important properties which we have anticipated.

- The length of the magnetization vector is preserved $d\boldsymbol{M}^2/dt = 2\boldsymbol{M} \cdot (d\boldsymbol{M}/dt) = -2\gamma \boldsymbol{M} \cdot (\boldsymbol{M} \times \boldsymbol{F}) = 0$.
- In the case of non-interacting magnetic moments, it reduces to the equation $d\boldsymbol{M}/dt = -\gamma \boldsymbol{M} \times \boldsymbol{H}_{\text{ext}}$ which describes precession of $\boldsymbol{M}$ around an external field.

We use the rationalized variables defined in (3.8) and measure lengths in exchange length units (3.9) to obtain the rationalized form of the Landau-Lifshitz equation

$$\frac{\partial \boldsymbol{m}}{\partial \tau} = -\boldsymbol{m} \times \boldsymbol{f}, \qquad \boldsymbol{f} = -\frac{\delta E}{\delta \boldsymbol{m}} = \Delta \boldsymbol{m} - q\, m_3 \hat{\boldsymbol{e}}_3 + \boldsymbol{h}_m + \boldsymbol{h}_{\text{ext}}. \tag{3.20}$$

We have introduced the rationalized time variable

$$\tau = \frac{t}{\tau_0}, \qquad \tau_0 = \frac{1}{\gamma \mu_0 M_s}, \tag{3.21}$$

that is, time is measured in units of τ_0. Since $\gamma\mu_0 = 2.21 \times 10^5\, \mathrm{m\, A^{-1} s^{-1}}$ we find for permalloy $\tau_0 = 6.56 \times 10^{-12}$ sec.

Note that static solutions of the LL equation satisfy

$$\boldsymbol{m} \times \boldsymbol{f} = 0. \tag{3.22}$$

Using δ as the unit of length the LL equation has the form

$$\frac{\partial \boldsymbol{m}}{\partial \tau} = -\boldsymbol{m} \times \boldsymbol{f}, \qquad \boldsymbol{f} = -\frac{\delta E}{\delta \boldsymbol{m}} = \Delta \boldsymbol{m} - m_3 \hat{\boldsymbol{e}}_3 + \frac{\boldsymbol{h}_m}{q} + \frac{\boldsymbol{h}_{\text{ext}}}{q}. \tag{3.23}$$

EXERCISE 3.4. *Show that energy* (3.10) *is conserved by the Landau-Lifshitz equation* (3.20).

3.4. Gilbert damping. While the formulation of the Landau-Lifshitz equation is hamiltonian and it thus conserves energy, it is clear that dissipation mechanisms are present in a magnetic material. The effect of dissipation is to damp the precessional motion of the magnetization. The common way to include dissipation in the LL equation in to add a term which represents *Gilbert* damping and thus obtain the following Landau-Lifshitz-Gilbert (LLG) equation:

$$\frac{\partial \boldsymbol{M}}{\partial t} = -\gamma \boldsymbol{M} \times \boldsymbol{F} + \frac{\alpha}{M_s} \boldsymbol{M} \times \frac{\partial \boldsymbol{M}}{\partial t}. \tag{3.24}$$

The constant α is dimensionless and it is called the *dissipation constant*. Typical values are $\alpha < 0.02$. The damping term in the above equation is constructed such that it is perpendicular to the magnetization (so it conserves its length) and it is also perpendicular to the precessional motion of $\boldsymbol{M}$ (so it damps this motion).

Using rationalized variables and units we obtain

$$\dot{\boldsymbol{m}} = -\boldsymbol{m} \times \boldsymbol{f} + \alpha\, \boldsymbol{m} \times \dot{\boldsymbol{m}}, \tag{3.25}$$

where we have adopted the short-hand notation $\dot{\boldsymbol{m}} = \partial \boldsymbol{m}/\partial \tau$. We may solve the above LLG equation for the $\partial \boldsymbol{m}/\partial \tau$ and obtain a form of the equation more convenient for calculations. First, take the cross product with $\boldsymbol{m}$ to obtain

$$\boldsymbol{m} \times \dot{\boldsymbol{m}} = -\boldsymbol{m} \times (\boldsymbol{m} \times \boldsymbol{f}) + \alpha\, \boldsymbol{m} \times (\boldsymbol{m} \times \dot{\boldsymbol{m}}) = -\boldsymbol{m} \times (\boldsymbol{m} \times \boldsymbol{f}) - \alpha\, \dot{\boldsymbol{m}}.$$

Substitute in the first equation to find the LLG equation in the form

$$\dot{\boldsymbol{m}} = -\alpha_1\, \boldsymbol{m} \times \boldsymbol{f} - \alpha_2\, \boldsymbol{m} \times (\boldsymbol{m} \times \boldsymbol{f}), \qquad \alpha_1 = \frac{1}{1+\alpha^2}, \quad \alpha_2 = \frac{\alpha}{1+\alpha^2}. \tag{3.26}$$

EXERCISE 3.5. *Show that energy* (3.10) *is continuously decreasing under the Landau-Lifshitz-Gilbert equation* (3.26) *for* $\dot{\boldsymbol{m}} \neq 0$.

4. Formulations for the magnetization

4.1. Angle variables. As the magnetization is a vector with length $\boldsymbol{m}^2 = 1$ it takes values on the unit sphere (S^2). Therefore it can be represented by two angles: $0 \le \Theta \le \pi$ and $0 \le \Phi < 2\pi$. The magnetization vector components are then given by the usual formulae familiar from spherical coordinate tranformations:

$$m_1 = \sin\Theta \cos\Phi, \quad m_2 = \sin\Theta \sin\Phi, \quad m_3 = \cos\Theta.$$

One should notice that we have expressed the three components of the vector $\boldsymbol{m}$ by only two angle variables. We have thus resolved the constraint on the magnetization vector.

The LL equation in the angle variables are written in the manifestly hamiltonian form:

$$\begin{aligned} \dot{\Theta} &= -\frac{1}{\sin\Theta} \frac{\delta E}{\delta \Phi} \\ \dot{\Phi} &= \frac{1}{\sin\Theta} \frac{\delta E}{\delta \Theta}. \end{aligned} \tag{4.1}$$

EXERCISE 4.1. *Derive Eq.* (4.1) *from the LL equation* (3.20). *[Hint: use the variables* $\Pi \equiv \cos\Theta,\ \Phi$.*]* □

EXERCISE 4.2. *Using the LLG equation* (3.24) *derive the corresponding equations for the variables* $\Pi = \cos\Theta,\ \Phi$*:*

$$\begin{aligned} \dot{\Pi} &= \alpha_1 \frac{\delta E}{\delta \Phi} - \alpha_2 \sin^2\Theta \frac{\delta E}{\delta \Pi} \\ \dot{\Phi} &= -\alpha_1 \frac{\delta E}{\delta \Pi} - \frac{\alpha_2}{\sin^2\Theta} \frac{\delta E}{\delta \Phi}. \square \end{aligned}$$

EXERCISE 4.3. *Write the exchange energy and the anisotropy energy in terms of* Θ, Φ*:*

$$\begin{aligned} E_{\rm ex} &= \frac{1}{2} \int \left[(\boldsymbol{\nabla}\Theta)^2 + \sin^2\Theta\, (\boldsymbol{\nabla}\Phi)^2 \right]\, d^3\boldsymbol{r} \\ E_{\rm a} &= \frac{q}{2} \int \cos^2\Theta\, d^3\boldsymbol{r}. \square \end{aligned}$$

4.2. Complex stereographic variable. Another method to resolve the constraint on the magnetization vector is to use a complex variable. Specifically, we define

$$\Omega = \frac{m_1 + i m_2}{1 + m_3}. \tag{4.2}$$

The variable Ω gives the stereographic projection of the vector $\boldsymbol{m}$ on the plane. Therefore there is a one-to-one correspondence between the vector $\boldsymbol{m}$ and Ω. We may invert Eq. (4.2) to find

$$m_1 = \frac{\Omega + \overline{\Omega}}{1 + \Omega\overline{\Omega}}, \quad m_2 = \frac{1}{i}\frac{\Omega - \overline{\Omega}}{1 + \Omega\overline{\Omega}}, \quad m_3 = \frac{1 - \Omega\overline{\Omega}}{1 + \Omega\overline{\Omega}},$$

where $\overline{\Omega}$ is the complex conjugate of Ω. In further calculations we may use as field variables either the real and imaginary parts of Ω or the two complex variables Ω and $\overline{\Omega}$. For an intuitive understanding of the significance of the values of Ω note the following

$$m_3 = 1 \Rightarrow \Omega = 0, \qquad m_3 = -1 \Rightarrow \Omega \to \infty, \qquad m_3 = 0 \Rightarrow |\Omega| = 1.$$

It takes some calculations to find the form of the LL equation (3.20) in terms of Ω:

$$i\,\dot{\Omega} = -\frac{1}{2}\,(1 + \Omega\overline{\Omega})^2\,\frac{\delta E}{\delta\overline{\Omega}}, \tag{4.3}$$

where E is the energy written in terms of $\Omega, \overline{\Omega}$. It is very convenient that the full LLG equation can be written in the form

$$(i + \alpha)\,\dot{\Omega} = -\frac{1}{2}\,(1 + \Omega\overline{\Omega})^2\,\frac{\delta W}{\delta\overline{\Omega}}. \tag{4.4}$$

Exercise 4.4. *Show that the stereographic variable can be written in term of the angles Θ, Φ as*

$$\Omega = \tan\left(\frac{\Theta}{2}\right)\,\exp(i\Phi).\Box$$

Exercise 4.5. *Write the exchange, anisotropy and external field energy in terms of Ω. These are*

$$E_{\text{ex}} = 2\int \frac{\partial_\mu\Omega\,\partial_\mu\overline{\Omega}}{(1 + \Omega\overline{\Omega})^2}\,d^3\boldsymbol{r} \tag{4.5}$$

$$E_{\text{a}} = \frac{q}{2}\int \left(\frac{1 - \Omega\overline{\Omega}}{1 + \Omega\overline{\Omega}}\right)^2\,d^3\boldsymbol{r}.\Box$$

Example 4.6.

Let us assume an external field of the form

$$\boldsymbol{h}_{\text{ext}} = (0, 0, h_3).$$

Then the external field energy can be written as

$$E_{\text{ext}} = -\int \boldsymbol{h}_{\text{ext}}\cdot\boldsymbol{m}\,d^3\boldsymbol{r} = -\int h_3\,\frac{1 - \Omega\overline{\Omega}}{1 + \Omega\overline{\Omega}}\,d^3\boldsymbol{r}.$$

We have

$$\frac{\delta E_{\text{ext}}}{\delta\overline{\Omega}} = \frac{2h_3}{(1 + \Omega\overline{\Omega})^2}\,\Omega.$$

The equation of motion for a *free* magnetic moment then reads

$$(i + \alpha)\,\frac{\partial\Omega}{\partial t} = -h_3\,\Omega$$
$$\Rightarrow \Omega(t) = \Omega_0\,e^{-\alpha_2 h_3}\,e^{i\alpha_1 h_3},$$

where $\Omega_0 = \Omega(t = 0)$ is the initial magnetization. The second factor in the product gives the dissipative effect, i.e., $\Omega(t \to \infty) \to 0$ (for $h_3 > 0$), while the last factor gives the effect from the conservative part of the equation, i.e., precession of the magnetization around the third axis, viewed here as precession of Ω on the complex plane. $\Box$

CHAPTER 2

Domain Walls

1. Landau-Lifshitz wall

For a magnetic sample it is reasonable to assume that it may be uniformly magnetized along a certain direction as the exchange interaction tends to align all spins to each other. It is though evident that the direction of uniform magnetization is arbitrary if the magnet is isotropic. Let us consider the case of uniaxial anisotropy as in Eq. (3.3). Both orientatons $\pm\hat{\boldsymbol{e}}_3$ are favoured (when the z-axis is the *easy* direction of the magnetization) and thus there are two degenerate ground states for the system, namely the uniform magnetization states, $\boldsymbol{m} = \pm(0,0,1)$. We further mention that, for uniform magnetization, the magnetostatic field $\boldsymbol{h}_{\rm m}$ given by Maxwell's equations (3.4) is zero, since $\boldsymbol{\nabla}\cdot\boldsymbol{m} = 0$. This is true as long as we suppose that the sample is infinite (extended in all space direction). If we would take into account the sample boundaries then these would impose boundary conditions on the equations which would give rise to some nonzero magnetostatic field $\boldsymbol{h}_{\rm m}$.

For a system with two degenerate ground states one can easily imagine that different regions of the sample may be in one or in the other ground state. We call the regions with uniform magnetization *magnetic domains*. Then the question arises what the magnetization is between these domains. For definiteness let us suppose that we have two domains which are magnetized along the z-axis while they are located at $x > 0$ and $x < 0$ respectively (they are separated by the yz-plane). Landau and Lifshitz have proposed that the magnetization rotates gradually in the yz-plane as we move from one domain to the other along the x-axis. We may then write the three components of the magnetization as

$$m_1 = 0, \qquad m_2 = \sin\Theta, \qquad m_3 = \cos\Theta, \tag{1.1}$$

which has the useful property that the magnetization is expressed via a single function $\Theta = \Theta(x)$. Since rotation is confined on the yz-plane we have $\boldsymbol{\nabla}\cdot\boldsymbol{m} = 0$ and thus no magnetostatic field is produced. We substitute the above form of the magnetization in the LL equation (3.23) and look for static solutions, so we find the equation

$$\partial_x^2\Theta - \sin\Theta\cos\Theta = 0. \tag{1.2}$$

Note that we have assumed easy-axis anisotropy as in Eq. (3.14) with $q = 1$. A more convenient way to derive the latter equation is to write first the energy in terms of Θ:

$$E = \frac{1}{2}\int(\partial_x\Theta)^2\,dx + \frac{1}{2}\int\sin^2\Theta\,dx$$

and then write $\delta E/\delta\Theta = 0$ to find Eq. (1.2).

This equation is integrated once to give

$$\partial_x\Theta = \pm\sin\Theta.$$

The solution of the latter is

$$e^{\pm x} = \tan(\Theta/2) \tag{1.3}$$

where the plus and minus correspond to the plus and minus in the differential equation.

From Eq. (1.3) using trigonometric identities we find the magnetization vector

$$m_1 = 0, \qquad m_2 = \pm\frac{1}{\cosh(x)}, \qquad m_3 = \pm\tanh(x) \tag{1.4}$$

where the $\pm$ can be taken in any combination. We restore the physical variables and units by the substitutions $\boldsymbol{m} = \boldsymbol{M}/M_s$ and $x \to x/\delta$, and have [**4**]

$$M_1 = 0, \qquad M_2 = \frac{M_s}{\cosh(x/\delta)}, \qquad M_3 = M_s \tanh(x/\delta).$$

This manifests that the domain wall width is $\delta = \sqrt{A/K}$.

The wall with the structure given in Eq. (1.4) is usually called a *"Bloch wall"*. A second type of domain wall which appears in practice, especially in very thin films, is similar in form to the wall described above but the rotation of the magnetization vector takes place in the xz-plane. That is, $m_2 = 0$ while m_1, m_3 would be non-zero. Such a domain wall produces a magnetostatic field (since $\boldsymbol{\nabla} \cdot \boldsymbol{m} = 0$). It is called the *"Neél wall"*.

EXERCISE 1.1. *Apply Derrick's scaling argument and find a relation between the exchange and anisotropy terms for the static domain wall.*

2. Propagating domain wall

Many interesting phenomena in magnetic materials refer to switching of the magnetization of domains. One way to achieve this is to shift the domain wall between two domains of opposite magnetization. This way one of the domains expands at the expense of the other. This brings forward the problem of a propagating domain wall.

Suppose two domains of opposite magnetization separated by a Landau-Lifshitz wall. If we apply an uniform external field

$$\boldsymbol{h}_{\text{ext}} = (0, 0, h_{\text{ext}})$$

then we may expect that expansion of the domain along the direction of the field may proceed through motion of the domain wall. We use Eq. (3.26), that is we suppose the presence of an external field as well as dissipation in the system [**4**]. We will be looking for a propagating wall of the special form

$$\boldsymbol{m} = \boldsymbol{m}(x - vt),$$

where v is a constant velocity for the domain wall.

Having in mind the structure of the static Landau-Lifshitz wall we will proceed with a modification of it in order to arrive at a structure which will be dynamic (propagating). Probably the simplest modification of the ansatz (1.1) is to assume a non-zero but still uniform Ψ. Then we have

$$\begin{aligned} &m_1 = \sin\Theta\,\cos\Phi, \qquad m_2 = \sin\Theta\,\sin\Phi, \qquad m_3 = \cos\Theta, \\ &\Phi : \text{const.}, \qquad \Theta = \Theta(x, t). \end{aligned} \tag{2.1}$$

We now have to consider the magnetostatic field created by such a magnetic configuration. We can argue that there is no magnetostatic field in the yz-plane because $\boldsymbol{m}$ does not depend on y and z. However, in the x direction we expect a nonzero component of the magnetostatic field. The Maxwell equation $\boldsymbol{\nabla} \cdot (\boldsymbol{h}_m + \boldsymbol{m}) = 0$ indicates that $\boldsymbol{h}_m = -m_1 \hat{\boldsymbol{e}}_1$, and this would be a good approximation if we suppose that the domain wall is confined in a thin layer. Putting together our remarks we have

$$f_1 = (m_1)'' - m_1, \qquad f_2 = (m_2)'', \qquad f_3 = (m_3)'' + q\, m_3 + h_{\text{ext}},$$

where the double prime denotes second derivative in space. The energy functional which gives the above $\boldsymbol{f} = -\delta E/\delta \boldsymbol{m}$ is

$$E = \frac{1}{2}\int \partial_x \boldsymbol{m}\cdot\partial_x \boldsymbol{m}\, dx + \frac{1}{2}\int \left[-q\,(m_3)^2 + (m_1)^2\right]\, dx - \int h_{\text{ext}} m_3\, dx.$$

It is convenient to use the angle variables in order to proceed further, so we have

$$E = \frac{1}{2}\int \left[(\partial_x\Theta)^2 + \sin^2\Theta(\partial_x\Phi)^2\right]\, dx + \frac{1}{2}\int \left[q\,\sin^2\Theta + \cos^2\Phi\sin^2\Theta\right]\, dx - \int h_{\text{ext}}\cos\Theta\, dx.$$

The equations of motion for Θ, Φ are

$$\sin\Theta(\dot{\Theta} + \alpha\dot{\Phi}) = -\frac{\delta E}{\delta\Phi}$$
$$\sin\Theta\dot{\Phi} - \alpha\dot{\Theta} = \frac{\delta E}{\delta\Theta}$$

and, since we have $\dot{\Phi} = 0 \Rightarrow \Phi = \Phi_0$, they reduce to

$$\dot{\Theta} = (\cos\Phi_0\sin\Phi_0)\sin\Theta$$
$$\alpha\dot{\Theta} = \partial_x^2\Theta - \epsilon^2\cos\Theta\sin\Theta - h_{\text{ext}}\sin\Theta, \qquad \epsilon^2 = q + \cos^2\Phi_0. \tag{2.2}$$

Let us consider the following ansatz

$$\tan\left(\frac{\Theta}{2}\right) = e^{\epsilon(x-vt)} \tag{2.3}$$

which represents a domain wall propagating with constant velocity v. For this specific ansatz we calculate

$$\frac{\partial}{\partial x}\tan\left(\frac{\Theta}{2}\right) = \epsilon\,\tan\left(\frac{\Theta}{2}\right), \qquad \frac{\partial}{\partial t}\tan\left(\frac{\Theta}{2}\right) = -\epsilon v\,\tan\left(\frac{\Theta}{2}\right).$$

So that

$$\frac{\partial}{\partial x}\tan\left(\frac{\Theta}{2}\right) = \frac{1}{2}\frac{\partial_x\Theta}{\cos^2\left(\frac{\Theta}{2}\right)} \Rightarrow \epsilon\,\tan\left(\frac{\Theta}{2}\right) = \frac{1}{2}\frac{\partial_x\Theta}{\cos^2\left(\frac{\Theta}{2}\right)} \Rightarrow \partial_x\Theta = \epsilon\,\sin\Theta.$$

Similarly

$$\dot{\Theta} = -\epsilon v\,\sin\Theta$$

and the second derivative

$$\partial_x^2\Theta = \epsilon\,\cos\Theta\partial_x\Theta \Rightarrow \partial_x^2\Theta = \epsilon^2\,\cos\Theta\sin\Theta.$$

Substitute the results in the Eqs. (2.2) to find that they are satisfied under the conditions

$$v = -\frac{\sin(2\Phi_0)}{2\epsilon} = \frac{h_{\text{ext}}}{\epsilon\alpha}. \tag{2.4}$$

These conditions link the domain wall velocity to the parameters of the system. One could think of h_{ext} as a free parameter. Given a specific material, with a certain dissipation constant α, we may tune h_{ext} in order to obtain the desired velocity. As we tune the parameters the angle Φ_0 of the domain wall is adjusted to appropriate values.

The formula for the velocity in (2.4) implies that a propagating wall with constant velocity is possible only for field values

$$h_{\text{ext}} \leq h_w, \qquad h_w \equiv \frac{\alpha}{2}$$

where h_w is called the *Walker field.*

EXAMPLE 2.1. *Plot the wall velocity as a function of the external field*

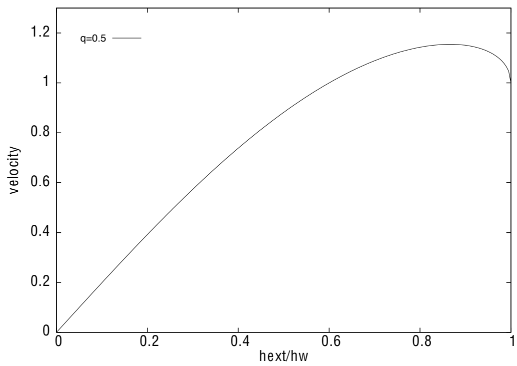

FIGURE 1. The velocity v of a Bloch wall relative as a function of the reduced external field $h_{\rm ext}/h_w$. We have used $q = 0.5$.

We may write

$$\frac{h}{h_w} = -\sin(2\Phi_0), \qquad \cos^2\Phi_0 = \frac{1 \pm \sqrt{1-(h_{\rm ext}/h_w)^2}}{2}.$$

The plus sign corresponds to a wall which is Neél ($\Phi_0 = 0, \pi$) for $h_{\rm ext} = 0$ and the minus sign corresponds to a wall which is Bloch ($\Phi_0 = \pi/2$) for $h_{\rm ext} = 0$. We have

$$v = \frac{h_{\rm ext}/h_w}{2\epsilon}, \qquad \epsilon = \left\{ q + \frac{1}{2}\left[1 \pm \sqrt{1-(h_{\rm ext}/h_w)^2}\right] \right\}^{1/2}.$$

The velocity v of a Bloch wall relative as a function of the reduced external field $h_{\rm ext}/h_w$ is shown in Fig. 1. The velocity at the maximum field h_w is

$$v_w = \frac{1}{2\sqrt{q+\frac{1}{2}}} \tag{2.5}$$

and it is called the *Walker limiting velocity.* The maximum velocity for a domain wall may be achieved for some field $h_{\rm ext} < h_w$ as shown in Fig. 1.

EXAMPLE 2.2. *Obtain the propagating domain wall solution using the LL equation for the magnetization vector.*

The LLG equation (3.25) has the form

$$\begin{aligned} vm_1' &= m_2(f_3 + \alpha v m_3') - m_3(f_2 + \alpha v m_2') \\ vm_2' &= m_3(f_1 + \alpha v m_1') - m_1(f_3 + \alpha v m_3') \\ vm_3' &= m_1(f_2 + \alpha v m_2') - m_2(f_1 + \alpha v m_1'). \end{aligned}$$

Going through calculations we should find that the domain wall profile is

$$m_1 = \frac{\cos\Phi_0}{\cosh(\sqrt{\epsilon}\,x)}, \qquad m_2 = \frac{\sin\Phi_0}{\cosh(\sqrt{\epsilon}\,x)}, \qquad m_3 = \tanh(\sqrt{\epsilon}\,x). \tag{2.6}$$

The third equation is satisfied if we impose the condition

$$v = -\frac{\sin(2\Phi_0)}{2\epsilon}, \qquad \epsilon^2 \equiv q + \cos^2\Phi_0.$$

Then the first two equations are satisfied for

$$v = \frac{h_{\text{ext}}}{\epsilon \, \alpha}.$$

CHAPTER 3

Magnetic Vortices

1. Introduction

1.1. Ordinary vortices in fluids. We know from common experience that vortices are pervasive in nature and they play a significant role in various physical systems. The most well known examples appear in fluids where fluid vortices play a central role in the description and understanding of the motion of fluids, including complicate dynamical phenomena such as turbulent flow.

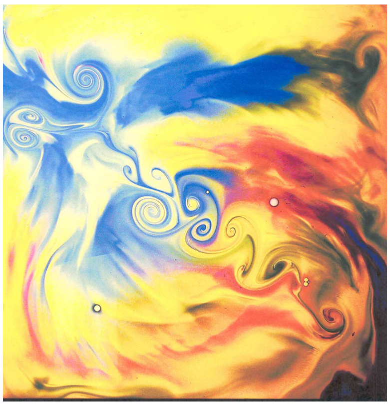

FIGURE 1. Photograph of vortices and vortex pairs which have been created by the motion of a particle on the surface of a fluid.

The description of the motion of fluids is based on non-linear partial differential equations. It is though possible to reduce these equations to much simpler forms if we want to describe, within some approximation, the motion of a vortex which is located away from other vortices. In that case we assume that the area of the vortex is small compared to the distance to other vortices. We then approximate the vortex position by a single point and we call this the *point vortex approximation*.

A significant quantity for the description of vortices is the local vorticity (γ) defined at every point of the fluid and it is the rotation of its velocity ($\boldsymbol{\nabla} \times \boldsymbol{v}$). The total vorticity is the integral of the vorticity over the area of the fluid.

$$\Gamma = \int \gamma dxdy$$

and it is considered as the strength of the vortex. It is interesting that it can be shown that the following are conserved quantities in vortex motion:

$$I_x = \int x\gamma\, dxdy, \qquad I_y = \int y\gamma\, dxdy. \tag{1.1}$$

It is evident that these quantities can be considered to give the position of a vortex (if we normalize by Γ). For example, in the case of point vortices, the above integrals just give the vortex position multiplied by its strength (total vorticity) [**8, 9**].

1.2. Quantized vortices in superfluids. Some fluids exhibit unusual properties when they are in very low temperatures. Probably their most impressive property is that they can flow without dissipation, that is without their motion be decelerated. Such fluids are called *superfluids*. Superfluidity was first observed in liquid helium in temperatures $T < 2.7$ Kelvin. More recently (1995) superfluid gases in the form of vapours of alkali metals (Li, Na, K, Rb, Cs) have been obtained and experimentally studied. Vapours of alkali metals are typically trapped by magnetic and optical (laser) fields and they are subsequently cooled by a series of techniques to temperatures $T \sim 10-100$ nanoKelvin. Such atomic gases are extremely dilute and a typical trap may contain $10^5 - 10^6$ atoms confined in spatial dimensions of the order of $10\,\mu$m.

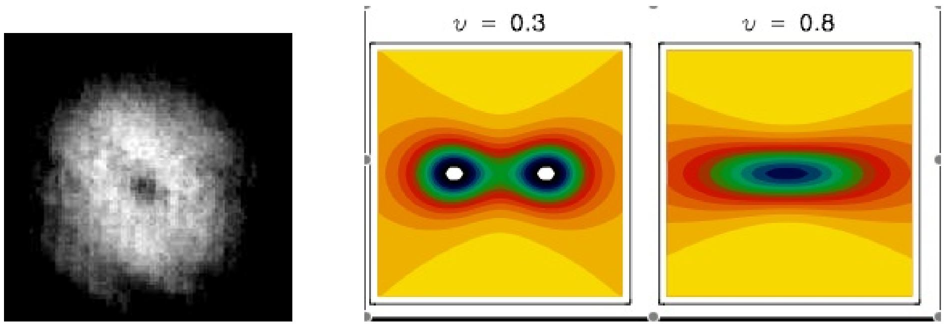

FIGURE 2. [Left:] Photograph of a superfluid vortex (white colour indicates high density and black indicates empty space). [Right:] Numerical simulation of a superfluid vortex pair (the density of the superfluid is given by a colour code).

An additional property of superfluids is that superfluid vortices have strength which may only be an integer multiple of a basic quantity and we call these *quantized vortices*. This property of vortices is related to their property of frictionless flow. Superfluids are studied using quantum mechanics. In some cases one can approximate superfluid dynamics by non-linear partial differential equations (Gross-Pitaevskii model). These equations differ significantly from those for ordinary fluids.

One important point is that the vorticity in superfluid is related to topological features of the field which describes the superfluid (i.e., the complex wavefunction which describes the superfluid). One can define a vorticity, where the total vorticity $\Gamma = \int \gamma dxdy$ takes only discrete values and it can be interpreted as a topological number. Conserved quantities formally identical to (1.1) exist also in the present case. Furthermore, the point vortex approximation can be employed in this system, too, and obtain simple differential equations for the vortex motion.

1.3. Magnetic vortices. Although we tend to link vortices with fluid motion, the case is that vortices appear in many systems which may not present physical fluid motion, as mathematical structures of vector fields. An interesting example are magnetic materials where we have *magnetic vortices*.

The microscopic structure in a magnetic material is described by the magnetization vector $\boldsymbol{m}(x, y)$. As interesting question is the following: what are the structures formed by the magnetization vector (which is actually a vector field) and what is their dynamics? The answers to such questions are important, for example, when we would like to store and retrieve information from a magnetic disc, since the information is stored as particular magnetization structures. We need to know the dynamics of such structures if we want to be able to change them in a controlled way.

Although there is no physical fluid flow in magnetic materials, we can define a quantity n which has properties corresponding to the fluid vorticity (γ). The magnetic vorticity n is related to topological features of the magnetization and the total vorticity $\mathcal{N} = \int n\, dxdy$ is an integer multiple of a basic quantity.

We finally mention that we have conserved quantities of the motion

$$I_x = \int xn\, dxdy, \qquad I_y = \int yn\, dxdy \tag{1.2}$$

which are formally similar to the conserved quantities for fluids in Eq. (1.1). It can also be shown that, if we make a point vortex approximation, the dynamics of magnetic vortices is modeled by equation similar to those for point fluid vortices.

2. Magnetic films

We consider a thin film and we suppose that the magnetization configuration does not vary significantly in the direction of the film thickness (assumed to be along the z-axis). We are thus motivated to study planar magnetic configurations, that is, the magnetization vector is considered a function of only two spatial variables and time $\boldsymbol{m} = \boldsymbol{m}(x, y, t)$.

2.1. Derrick's scaling argument. Let us suppose a simple model for a magnet where only the exchange and an anisotropy term are present, so the energy is

$$E = E_{\rm ex} + E_{\rm a}.$$

The first question to ask is whether this model has any static solutions. An argument due to Derrick starts by supposing the existence of a static solution $\boldsymbol{m}(x, y)$ [**12**]. Such a static solution would have to be a minimum of the energy functional. Therefore, any deformation of such a configuration would give a higher energy state. This holds in particular for any of the static solution. Specifically, suppose that we scale both spatial coordinates by a factor λ and we obtain the dilated (or shrinked) configuration $\boldsymbol{m}_\lambda = \boldsymbol{m}(\lambda x, \lambda y)$, then this would have energy

(2.1)
$$E_{\rm ex}(\lambda) = \frac{1}{2}\int \partial_\mu \boldsymbol{m}(\lambda x, \lambda y)\cdot\partial_\mu \boldsymbol{m}(\lambda x, \lambda y)\, d^2x = \frac{1}{2}\int \partial_\mu \boldsymbol{m}(x', y')\cdot\partial_\mu \boldsymbol{m}(x', y')\, d^2x' = E_{\rm ex}(\lambda = 1)$$

(2.2)
$$E_{\rm a}(\lambda) = \frac{q}{2}\int [m_3(\lambda x, \lambda y)]^2\, d^2x = \frac{1}{\lambda^2}\frac{q}{2}\int [m_3(x', y')]^2\, d^2x' = \frac{1}{\lambda^2}\, E_{\rm a}(\lambda = 1).$$

By defining variables $x' = \lambda x,\ y' = \lambda y$, the integrals have been transformed to those for $\lambda = 1$, that is, for the solution with the minimum energy. Derrick's argument indicates that we should write the condition for the energy to be minimum at $\lambda = 1$:

$$\frac{dE}{d\lambda}\Big|_{\lambda=1} = 0 \Rightarrow \frac{dE_{\rm ex}}{d\lambda}\Big|_{\lambda=1} + \frac{dE_{\rm a}}{d\lambda}\Big|_{\lambda=1} = 0 \Rightarrow \left[-\frac{2}{\lambda^3}E_{\rm a}\right]_{\lambda=1} = 0 \Rightarrow E_{\rm a}(\lambda = 1) = 0. \tag{2.3}$$

The result is that any static solution of the model would have to have zero anisotropy energy, which means $m_3 = 0$. This necessarily implies that the static configuration is uniform, i.e., the ground state of the ferromagnet. In conclusion, Derrick's argument gives a stringent enough condition to exclude, within the present model, the existence of any nontrivial static solutions.

In the next sections we will study static solutions in ferromagnetic films, which actually exist, but the corresponding models will be formulated such that they evade the above Derrick's result.

3. A vortex and the winding number

3.1. Easy-plane ferromagnets: ground state. Let us consider the case of easy-plane anisotropy (3.13). We further suppose, for simplicity, that the sample is infinite (it has no boundaries). While the magnetization is expected to avoid alignment with the z-axis, any direction of the magnetization in the xy-plane gives zero anisotropy energy. In other words, any uniform configuration of the form

$$\boldsymbol{m}_0 = (m_1, m_2, 0), \qquad m_1^2 + m_2^2 = 1,$$

where m_1, m_2 are constants, is a ground state with the same energy.

3.2. Axially symmetric vortex. For any nontrivial (nonuniform) magnetic configuration we would require that $\boldsymbol{m}(|\boldsymbol{r}| \to \infty) \to \boldsymbol{m}_0$, that is, the magnetization tends to a ground state configuration at spatial infinity. This is because, otherwise the magnetic configuration would have infinite energy and it would be unstable. On the other hand, any nontrivial configuration for which the magnetization goes to a certain uniform magnetization at spatial infinity is excluded by Derrick's argument.

In order to construct a nontrivial magnetic configuration we will exploit the infinite degeneracy of the ground state in-plane configurations. We impose the boundary condition at spatial infinity

$$m_1 + i m_2 = e^{i(\phi+\phi_0)}, \qquad m_3 = 0, \qquad \text{for } \rho \to \infty \tag{3.1}$$

where (ρ, ϕ) are polar coordinates and ϕ_0 is a constant. This boundary condition imposes that the magnetization points at a direction which rotates as we rotate around the origin, in other words, the magnetization vector has a simple dependence on ϕ.

A configuration which is consistent with the boundary condition (3.1) is given by the following axially symmetric ansatz, using the angle variables:

$$m_1 + i m_2 = \sin\Theta(\rho)\, e^{i(\phi+\phi_0)}, \qquad m_3 = \cos\Theta(\rho). \tag{3.2}$$

We have assumed that $\Theta = \Theta(\rho)$ (i.e., axial symmetry) while we implicitly set the angle variable $\Phi = \phi + \phi_0$.

The energy is written as

$$E = \frac{1}{2} \int_0^\infty \left[\left(\frac{\partial \Theta}{\partial \rho} \right)^2 + \frac{\sin^2\Theta}{\rho^2} + q\, \cos^2\Theta \right] (2\pi \rho d\rho). \tag{3.3}$$

The equation for a static vortex is given by $\delta E/\delta\Theta = 0$. The profile of an axially symmetric vortex found numerically is shown in Fig. 3. We only need to give $\Theta = \Theta(\rho)$ so that the axially symmetric vortex profile is fully specified. Note that the constant angle ϕ_0 drops out of the final equation and thus the solution is independent of it.

It is obvious that we expect $\Theta(\rho \to \infty) = \pi/2$ (as verified in Fig. 3), so that the boundary condition (3.1) is satisfied. On the other hand, we should also require $\Theta(\rho = 0) = 0$ or π, since otherwise the second term in the energy (3.3) would diverge. In Fig. 3 we have made the choice $\Theta(\rho = 0) = 0$ which means $m_z(\rho = 0) = 1$. We thus see that the magnetization in the central region of the vortex points "up". We usually call the central region of the vortex the *vortex core* and say that such a vortex as in Fig. 3 has positive polarity (we denote this polarity with $\lambda = +1$). Since the equations are symmetric with respect to the transformation $m_z \to -m_z$ (note the symmetry $\Theta \to \pi - \Theta$ in the energy (3.3)) we conclude that there is a static vortex solution of the equations with negative m_z (or $m_z \to -m_z$) and we say that this vortex has polarity $\lambda = -1$.

EXERCISE 3.1. *Solve numerially the equation for the axially symmetric vortex ansatz for the angle variable* $\Theta(\rho)$. *Then plot Fig. 3.* □

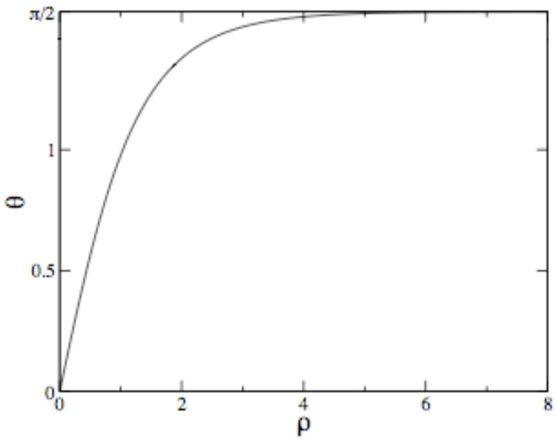

FIGURE 3. Static vortex profile calculated numerically: The angle Θ as a function of distance from the vortex center ρ for a static vortex.

EXAMPLE 3.2. *Estimate the energy of a vortex.*

It is instructive to calculate the vortex energy separately for the area of the vortex core and for the area away from it. Suppose that the vortex core lies inside a circle with radius R_c. Then define

$$E_c = \frac{1}{2}\int_0^{R_c} \left[\left(\frac{\partial \Theta}{\partial \rho} \right)^2 + \frac{\sin^2 \Theta}{\rho^2} + q \cos^2 \Theta \right] (2\pi\rho d\rho).$$

We have seen that $\sin\Theta(\rho=0)=0$ and we further suppose that $\sin^2\Theta(\rho=0)/\rho^2$ goes to a finite value as $\rho \to 0$. Then the energy E_c is finite and may only be calculated numerically. Let us also consider the energy of the vortex away from the vortex center, that is in the region where $\Theta = \pi/2$. An approximate form for this part of the energy is

$$E_v = \frac{1}{2}\int_{R_c}^{R} \frac{2\pi d\rho}{\rho} = \pi \ln\left(\frac{R}{R_c}\right).$$

We have integrated from R_c to a radius R (which may be thought of as the sample size), while we see that the energy diverges to infinity if we let $R \to \infty$. We thus note that the vortex energy is infinite in an infinite medium and it diverges logarithmically with the system size. □

EXAMPLE 3.3. *Write the equations for the vortex profile using the magnetization vector* $\boldsymbol{m}$*.*

A configuration which is consistent with the boundary condition (3.1) is given by the following axially symmetric ansatz:

$$m_1 + i m_2 = m_\perp(\rho)\, e^{i(\phi+\phi_0)}, \qquad m_3 = m_z(\rho). \tag{3.4}$$

Of course, we require $m_\perp^2 + m_z^2 = 1$, so we only need to find one of the two magnetization components. In polar coordinates we have the effective field

$$f_1 + i f_2 = f_\perp(\rho)\, e^{i(\phi+\phi_0)}, \qquad f_3 = f_z(\rho)$$

with

$$f_\perp = \Delta m_\perp - \frac{m_\perp}{\rho^2}$$
$$f_z = \Delta m_z - q m_z,$$

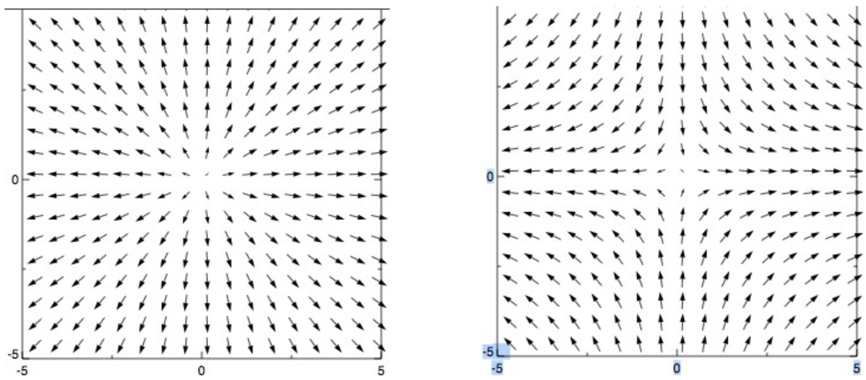

FIGURE 4. (Left) A vortex configuration ($S = 1$) with $\phi_0 = 0$. (Right) An antivortex configuration with ($S = -1$) $\phi_0 = 0$. We plot the projection of the magnetization on the plane: (m_1, m_2). The magnetization in the center of the figure is supposed to point either "up" or "down', that is, $\boldsymbol{m} = (0, 0, \lambda)$ where $\lambda = \pm 1$ is the vortex polarity.

and where

$$\Delta = \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho}$$

is the Laplacian whose form incorporates the symmetries of the present problem. We substitute the above results in the two first components of the LL equation which reduce to

$$m_z f_\perp - m_\perp f_z = 0.$$

The third component of the LL equation is identically satisfied.

An important remark is that we expect $m_\perp(\rho = 0) = 0$ due to the presence of the term $m_\perp/\rho^2$ in the effective field $f_\perp$. One can also show that the energy of a configuration with $m_\perp(\rho = 0) \neq 0$ would diverge. □

EXERCISE 3.4. *Suppose a sample in the shape of a ring. Find a vortex solution of the equations. Calculate the exchange and anisotropy energy. Calculate the magnetostatic field produced by the vortex.* □

3.3. Winding number. The most important feature of the vortex solution is that the magnetization has a different orientation in the xy-plane for different locations in space, even when we are away from the vortex center. More specifically, we can choose a circle with its center at the vortex core and we go around the circle registering the in-plane component of the magnetization (m_1, m_2) at each point. One should note that the in-plane magnetization vector can be characterized by the single angle Φ, that is, every vector orientation corresponds to a point on a circle. Therefore, as we go around a vortex rotating, say, anticlockwise on a circle in physical space, we measure an angle Φ for the magnetization. In this way we can define a mapping from the physical space to the magnetization space, this being a mapping from a circle to a circle.

In the case of the vortex presented in the previous subsection a full rotation (anticlockwise) around the vortex center gives a corresponding full rotating (again anticlockwise) of the magnetization vector on the xy-plane, or $\Delta\Phi = 2\pi$. We assign to the vortex a *winding number* S which represents this particular vortex feature. We define $S = \Delta\Phi/2\pi$, so the single full rotation of the magnetization vector is denoted by saying the $S = 1$. The magnetization vector on the plane (m_1, m_2) for a vortex is plotted in Fig. 4 for $\phi_0 = 0$, and in Fig. 5 for a vortex with $\phi_0 = \pi/2$.

It is evident that as we go around a full circle in physical space, physical quantities must be the same when we return to the initial point, therefore for the difference of the angle Φ of

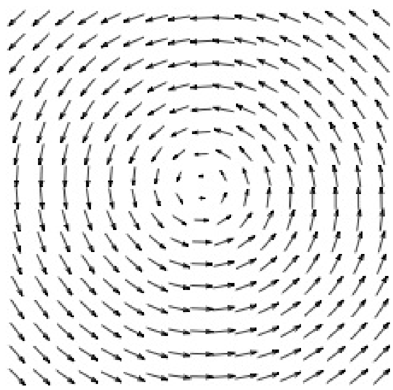

FIGURE 5. A vortex configuration ($S = 1$) with $\phi_0 = \pi/2$. We plot the projection of the magnetization on the plane: (m_1, m_2).

the magnetization we have $\Delta\Phi = (2\pi)S$ with $S = 0, \pm1, \pm2, \ldots$. We assume the following more general vortex ansatz:

$$m_1 + im_2 = \sin\Theta(\rho)\, e^{iS(\phi+\phi_0)}, \qquad m_3 = \cos\Theta(\rho).$$

The case $S = 1$ corresponds to the vortex discussed above, while in the case $S = -1$ we will call the configuration an *antivortex* (Fig. 4). The equation satisfied by the angle Θ for the antivortex is identical to that for the vortex and therefore the antivortex profile coincides with that of the vortex shown in Fig. 3. As a consequence, the antivortex energy (3.3) (when we take into account exchange and anisotropy only) is identical to that of the vortex.

The case $S = \pm2$ gives an object which may be called a double vortex or antivortex. We can apply the same methods as for the vortex in order to find the profile of a double vortex and its energy. However, it turns out that this is typically an unstable configuration which tends to split into two separate vortices. This is apparently due to the exchange energy which is approximately proportional to S^2 so that the energy of a double vortex is higher that that of two single vortices.

The winding number S is a topological invariant which can be defined as the degree of a mapping from the circle to the circle [**11, 12**]. It is for this reason that it may only take integer values. The winding number cannot change during the motion of the system because this would imply a discontinuous change of the magnetization configuration. Furthermore, S is a conserved quantity, that is, it remains constant under the dynamics prescribed by the model. It is not important to specify what the particular dynamics is, it suffices that this be continuous.

CHAPTER 4

Magnetic Bubbles

1. Magnetic bubbles

1.1. Perpendicular anisotropy. Let us consider a material with uniaxial anisotropy where the easy axis is perpendicular to the magnetic film, say, along the z axis. The anisotropy energy can then be taken to have the form (3.14), repeated here for convenience:

$$E_{\rm a} = \frac{q}{2} \int [(m_1)^2 + (m_2)^2]\, d^3\boldsymbol{r}.$$

If we assume only exchange and anisotropy energy in the system, it is straightforward to see that there are two degenerate ground states $\boldsymbol{m} = (0, 0, \pm 1)$. We may further see that Derrick's argument applies here, too. Derrick relation (2.3) is satisfied by the ground state solutions, and it excludes any other nontrivial static solution for the system.

1.2. A magnetic bubble. In experiments in the 1960s perpendicular anisotropy materials were used where one typically observes stripe domains at remanence [**10**]. The stripes point either "up" or "down" along the easy axis, and they are apparently separated by domain walls. An external magnetic field is typically applied which is perpendicular to the film and it thus favours one of the two easy-axis orientations of the magnetization. If we assume that the external magnetic field is uniform and it points to the positive z axis then the domains of "up" magnetization will expand at the expense of the oppositely magnetized domains. An interesting observation is that at relatively high fields the sample is saturated along the magnetic field, however, there remain some spots of opposite ("down") magnetization. A sketch of the process is shown in Fig. 1. In the same figure we give a sketch of the magnetic bubble which shows the bubble as a circular domain of magnetization opposite to the rest of the sample.

In order to study the creation of the bubble and the reasons for its appearing as a static magnetic configuration we should have a realistic model for the magnetic energy. Except for the exchange and the anisotropy contributions one should take into account the magnetostatic

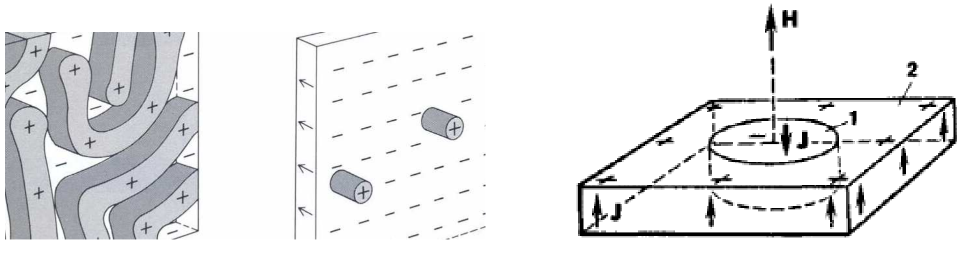

FIGURE 1. (Left) A perpendicular anisotropy material typically presents stripe domains at remanence. (Middle) On the application of an external magnetic field the sample is saturated along the direction of the field, however, some spots of opposite magnetization persist even at higher fields. [http://www.almasiconsulting.com/bubbles/bubbles.html] (Right) A sketch of a magnetic bubble. The bubble region is oppositely magnetized compared to the rest of the sample. [http://encyclopedia2.thefreedictionary.com/Magnetic+Bubble]

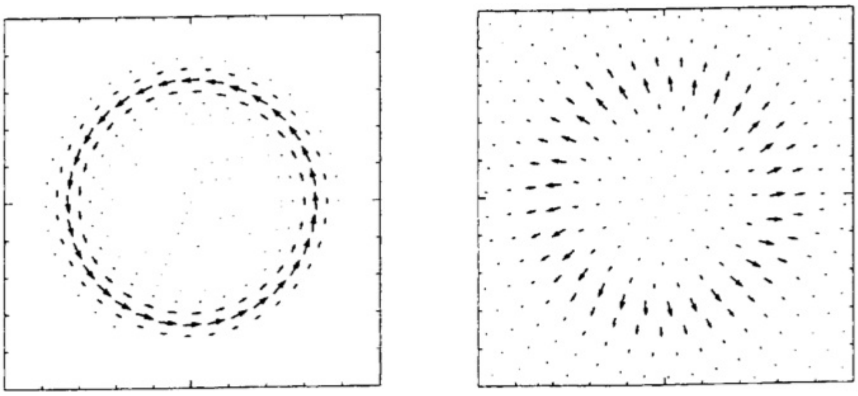

FIGURE 2. (Left) A magnetic bubble configuration. The projection of the magnetization of the plane is shown (m_1, m_2), which depicts the domain wall between the circular bubble region (magnetized, say, "down") and the outer region (magnetized "up"). The domain wall is a Bloch-type wall. (Right) Similar as in the left entry, but the domain wall is a Neél-type wall.

energy (3.5) which is particularly significant in thin films which are perpendicularly magnetized. This is because the orientation of the magnetic moments perpendicular to the film surface gives rise to free magnetic poles at the surface. The external magnetic field energy (3.6) should, of course, also be taken into account. Also note the important fact that the film, although thin, has a finite thickness which is important for the generation of the magnetostatic field. Thus, we need to study a three-dimensional model (not a two-dimensional model as in the case of vortices in the previous chapter). It can be shown that a procedure corresponding to Derrick's leads to the following relation which should be satisfied by all static solutions of the model [**6**]:

$$E_{\mathrm{ex}} + 3[E_{\mathrm{a}} + E_{\mathrm{ext}} + (E_{\mathrm{m}} - E_{\mathrm{m}}^{(0)})] = sd, \tag{1.1}$$

where $E_{\mathrm{m}}^{(0)}$ is the magnetostatic energy for uniform perpendicular magnetization, and d is the film thickness, while s is a surface integral on the film surface. It may sometimes be useful to check numerically that this relation is indeed satisfied by bubble solutions.

While the idea of a bubble domain of opposite magnetization (as in Fig. 1) seems simple, one should pay special attention to the thin circular layer separating the bubble domain from the outer domain. By numerical simulations we obtain, e.g., the examples shown in Fig. 2, where the magnetization vector rotates anticlockwise as we go around the bubble in the anticlockwise sense. This is a type of *axially (or cylindrically) symmetric* wall. When the magnetization is perpendicular to the radial direction we may call this a Bloch-type wall (left entry of figure), while when the magnetization is along the radial direction we may call this a Neél-type wall (right entry).

1.3. The Skyrmion number. We have seen that the configuration for a magnetic bubble is nontrivial, particularly at the domain wall around the bubble domain. In order to develop a systematic approach for the description of possible bubble configurations, we start by noting the basic fact that the magnetization vector is always pointing on a sphere of unit radius. This is only another way of saying that $\boldsymbol{m}$ has constant length equal to unity. We also note that the boundary condition for perpendicular anisotropy films is that the magnetization vector necessarily points along the z axis at spatial infinity, e.g., $\boldsymbol{m}(|\boldsymbol{r}| \to \infty) = (0, 0, 1)$, in other words $\boldsymbol{m}$ point to the north pole of the sphere. In particular, for bubble solutions, the magnetization points to the south pole of the sphere $\boldsymbol{m}(\boldsymbol{r} = 0) = (0, 0, -1)$ at the bubble center. Under the assumption of a smooth magnetization configuration, it is evident that $\boldsymbol{m}$ covers parts of the

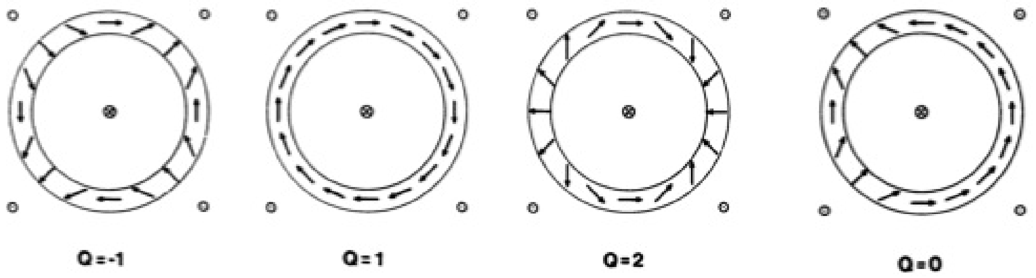

FIGURE 3. Schematic representation of various types of domain walls separating the bubble from the region of uniform magnetization. The symbol Q denotes the skyrmion number. (see Ref. [**5**])

sphere at intermediate points between $\boldsymbol{r}=0$ and ∞. For example, the magnetization points on the equator of the sphere at the domain wall (it fully covers the equator, as seen in both entries in Fig. 2).

Possibilities for different domain walls in bubbles are sketched in Fig. 3. At the domain wall we have the possibilities that the equator is covered once or more than once (twice, etc). It may also happen that as we go around the center clockwise, the equator is covered either clockwise or anticlockwise, and the latter possibility is denoted by a negative integer number. A further possibility is that the equator is partly covered in one sense and partially in the opposite sense thus giving an overall zero for the rotation angle of the magnetization. These possibilities imply that we can assign some topological number to the bubble configuration.

In order to identify the appropriate topological number, a crucial point is that, since the magnetization is considered uniform at spatial infinity, we can treat spatial infinity on the plane as a single point. Then the plane is isomorphic to a sphere. We can thus see that $\boldsymbol{m}(x,y)$ defines a mapping from the plane to a sphere, or, equivalently, from the sphere to the sphere ($S^2 \to S^2$). It is known that we can define classes of such mappings or magnetization configurations where a configuration in each class cannot be continuously deformed in a configuration of another class [**11**]. There is a discrete number of such classes and each is characterized by a topological number which takes integer values $\mathcal{N}=0,\pm1,\pm2,\ldots$. This will be called the *skyrmion number*.

In order to find a formula for the skyrmion number we should just use the Jacobian of transformation (n) from the plane to the sphere and integrate over the plane [**12**]

$$\mathcal{N}=\frac{1}{4\pi}\int n\,d^2x, \qquad n=\frac{1}{2}\epsilon_{\mu\nu}(\partial_\nu\boldsymbol{m}\times\partial_\mu\boldsymbol{m})\cdot\boldsymbol{m}, \qquad \mu,\nu=1,2. \tag{1.2}$$

The integrant n is called the local topological *density*, and the integrated quantity $\mathcal{N}$ takes integer values. This means that the sphere for the magnetization vector is covered an integer number of times. The sign of $\mathcal{N}$ denotes conventionally the sense of the rotation of the magnetization. The skyrmion number $\mathcal{N}$ for various bubble configurations is shown in Fig. 3 (where, unfortunately, skyrmion number is denoted by Q).

EXAMPLE 1.1. *Derive a formula for the skyrmion number for an axially symmetric configuration:*

$$m_1+im_2=[m_\rho(\rho)+im_\phi(\rho)]\,e^{i\phi}, \qquad m_3=m_z(\rho).$$

We first note that

$$n=(\partial_2\boldsymbol{m}\times\partial_1\boldsymbol{m})\cdot\boldsymbol{m}=\frac{1}{\rho}(\partial_\phi\boldsymbol{m}\times\partial_\rho\boldsymbol{m})\cdot\boldsymbol{m}.$$

For the axially symmetric configuration

$$n = \frac{1}{\rho}\,\left[(m_1 \partial_\phi m_2 - m_2 \partial_\phi m_1)\, \partial_\rho m_3 + (\partial_\phi m_1 \partial_\rho m_2 - \partial_\phi m_2 \partial_\rho m_2)\, m_3 \right].$$

We find

$$m_1 \partial_\phi m_2 - m_2 \partial_\phi m_1 = \mathrm{Im}\, \{ \partial_\phi (m_1 + i m_2)\, (m_1 - i m_2) \} = \ldots = m_\rho^2 + m_\phi^2$$
$$\partial_\phi m_1 \partial_\rho m_2 - \partial_\phi m_2 \partial_\rho m_2 = \mathrm{Im}\, \{ \partial_\phi (m_1 + i m_2)\, \partial_\rho (m_1 + i m_2) \} = -(m_\rho \partial_\rho m_\rho + m_\phi \partial_\phi m_\phi) = m_z \partial_\rho m_z.$$

We substitute and find the formula

$$n = \frac{1}{\rho} \frac{\partial m_z}{\partial \rho},$$

and we finally have

$$\mathcal{N} = \frac{1}{2} \int_0^\infty \frac{\partial m_z}{\partial \rho}\, d\rho = \frac{1}{2} \left[m_z(\rho = \infty) - m_z(\rho = 0) \right]. \quad \Box \tag{1.3}$$

EXERCISE 1.2. *Derive the formula for the skyrmion number using the complex variable* Ω*:*

$$\mathcal{N} = \frac{1}{4\pi} \int n\, d^2 x, \qquad n = 4\, \frac{|\Omega_{\bar z}|^2 - |\Omega_z|^2}{(1 + \Omega \overline{\Omega})^2} = 2i\, \epsilon_{\mu\nu}\, \frac{\partial_\nu \Omega\, \partial_\mu \overline{\Omega}}{(1 + \Omega \overline{\Omega})^2},$$

where $z = x + iy$. $\Box$

Bubble configurations can be conveniently written using the complex variable Ω. We define complex coordinates on the plane

$$z = x + iy, \qquad \bar z = x - iy,$$

so, in general, $\Omega = \Omega(z, \bar z)$. Let us make the simple choice $\Omega = \Omega(z)$, and take the particular example

$$\Omega = i\, \frac{a}{\bar z} = i\, \frac{\rho}{a}\, e^{i\phi} \Rightarrow m_\rho = 0, \quad m_\phi = \frac{2a\rho}{\rho^2 + a^2}, \quad m_z = \frac{\rho^2 - a^2}{\rho^2 + a^2}, \tag{1.4}$$

where a is a constant. We see that $m_z(\rho = 0) = -1$ and $m_z(\rho \to \infty) = 1$, that is, this form indeed represents a magnetic bubble. The component of the magnetization on the xy plane points azimuthally and it has significant values at $\rho \sim a$, while at $\rho = 0, \infty$ is vanishes. We therefore have a Bloch-type domain wall at $\rho \sim a$. Eq. (1.3) gives for this bubble $\mathcal{N} = 1$.

As another example suppose

$$\Omega = i\, \frac{\bar z}{a} = i\, \frac{a}{\rho}\, e^{-i\phi} \Rightarrow m_1 = \frac{2ay}{a^2 + \rho^2}, \quad m_2 = \frac{2ax}{a^2 + \rho^2}, \quad m_3 = \frac{a^2 - \rho^2}{a^2 + \rho^2}. \tag{1.5}$$

This configuration has the form of a magnetic bubble ($m_z(\rho = 0) = -1$ and $m_z(\rho \to \infty) = 1$) but the domain wall, which is at $\rho \sim a$, is of the type shown in the first entry on the left in Fig. 3. Such a bubble has $\mathcal{N} = -1$ and it has similarities to an antivortex.

We may combine the above two examples and study the following form

$$\Omega = i\, \frac{\bar z + a}{\bar z - a} = \frac{\rho^2 - a^2 + 2iay}{\rho^2 + a^2 - 2ax}. \tag{1.6}$$

Note that

$$\Omega \overline{\Omega} = \frac{\bar z + a}{\bar z - a}\, \frac{z + a}{z - a} = \frac{\rho^2 + a^2 + 2ax}{\rho^2 + a^2 - 2ax},$$

and find

$$m_1 = \frac{\rho^2 - a^2}{\rho^2 + a^2}, \quad m_2 = \frac{2ay}{\rho^2 + a^2}, \quad m_3 = -\frac{2ax}{\rho^2 + a^2}.$$

The following remarks will help to understand the type of configuration in Eq. (1.6).

(a) We have $m_1(\rho \to \infty) = 1$, that is the magnetization points in the x axis at spatial infinity.
(b) For $\bar{z}$ close to a we may write $\bar{z} \approx a + \bar{\zeta}$ and thus configuration (1.6) is similar to (1.4) (for $z \to \zeta$ and $a \to 2a$). Also, $\Omega(z = a) = \infty$ or $m_3 = -1$.
(c) Similarly, for $\bar{z}$ close to $-a$ we may write $\bar{z} \approx -a + \bar{\zeta}$ and thus configuration (1.6) is similar to (1.5). Also, $\Omega(z = a) = 0$ or $m_3 = 1$.

The conclusion is that configuration (1.6) is vortex-like close to $z = a$ with negative polarity, and it is antivortex-like close to $z = -a$ with positive polarity. Therefore we have a vortex-antivortex pair where the vortex and the antivortex have opposite polarity. We can find that it has a skyrmion number $\mathcal{N} = 1$.

We finally note that forms of the type $\Omega = \Omega(z)$ or $\Omega = \Omega(\bar{z})$ are solutions of the pure exchange model and we refer the reader elsewhere for this subject [**12**].

EXAMPLE 1.3. *We may show that for an axially symmetric vortex we have*

$$\mathcal{N} = -\frac{1}{2} S\lambda. \quad \square$$

EXERCISE 1.4. *Construct a vortex-antivortex pair with same polarity, using the variable Ω. Show that it has $\mathcal{N} = 0$.*

CHAPTER 5

Conservation laws

1. Introduction

Studying a system of equations does not necessarily mean finding its solutions. A very useful tool for a qualitative, and often quantitative, study are conservation laws, that is, relations giving quantities which are conserved by the equations of motion. Of course, all solutions of the equations must satisfy these relations.

We will typically study in this chapter the conservative LL equation in two space dimensions and suppose that the effective field contains the exchange and anisotropy interactions and possibly an external magnetic field

$$\frac{\partial \boldsymbol{m}}{\partial \tau} = -\boldsymbol{m} \times \boldsymbol{f}, \qquad \boldsymbol{f} = \Delta \boldsymbol{m} - q\, m_3 \hat{\boldsymbol{e}}_3 + \boldsymbol{h}_{\text{ext}}. \tag{1.1}$$

2. Total magnetization

As the equation presents an anisotropy with respect to the third direction in magnetization space, let us study the third component of the magnetization. It is reasonable that we study an integrated quantity, as is the total magnetization in the third direction

$$\mu = \int m_3\, d^2x. \tag{2.1}$$

In order to identify a conserved quantity related to the magnetization we may take the time derivative of the total magnetization:

$$\begin{aligned}\dot{\mu} &= \int \dot{m}_3\, d^2x = -\int (\boldsymbol{m} \times \boldsymbol{f})_3\, d^2x = -\int (\boldsymbol{m} \times \partial_\mu \partial_\mu \boldsymbol{m} + \boldsymbol{m} \times \boldsymbol{h})_3\, d^2x \\ &= -\int \partial_\mu (\boldsymbol{m} \times \partial_\mu \boldsymbol{m})_3 - \int (\boldsymbol{m} \times \boldsymbol{h})_3\, d^2x.\end{aligned}$$

We have used that the anisotropy term does not enter in the equation of motion for m_3. The first term on the right-hand-side is in the form of a total derivative. This term is an integral over the whole plane, and by the Divergence Theorem it gives a line integral over the boundary of the space of integration

$$\int_S \int \partial_\mu (\boldsymbol{m} \times \partial_\mu \boldsymbol{m})_3\, d^2x = \oint_{\partial S} (\boldsymbol{m} \times \partial_\mu \boldsymbol{m})_3\, d\boldsymbol{\ell}$$

where S is the surface of integration and ∂S is its line boundary. In our case, where we integrate over the infinite plane, we may take the boundary at infinity. If we suppose that the integrand in the line integral falls rapidly enough as we go to spatial infinity, then the line integral is zero. Let us take

$$\boldsymbol{h}_{\text{ext}} = (h_1, h_2, h_3)$$

so we finally have

$$\dot{\mu} = -\int (m_1 h_2 - m_2 h_1)\, d^2x. \tag{2.2}$$

An interesting case is when we have no external field, where we have the conservation law $\dot{\mu} = 0 \Rightarrow \mu =$ constant. That is, in the absence of an external field, the total magnetization must be conserved for all solutions of the equations of motion. We could for example suppose a vortex which was probed by some external field and this was then switched-off. The vortex will probably perform oscillations, but these should be such that the total magnetic moment of the vortex core remains constant. Certainly, if dissipation is present the oscillations would be damped and they will eventually stop, at the same time μ will change and converge to the value for the static vortex or bubble.

For a magnetic bubble, μ as defined in (2.1) would be infinite (in an infinite film). It is in this case more useful to define

$$\mu = \int (m_3 - 1)\, d^2x \tag{2.3}$$

where we suppose that $m_3 = 1$ is the magnetization at spatial infinity (away from the bubble). This quantity is finite and it is indeed conserved.

3. Linear momentum

A straightforward calculation gives the following useful relation [**5**]

$$\dot{n} = -\epsilon_{\mu\nu}\, \partial_\mu \boldsymbol{f} \cdot \partial_\nu \boldsymbol{m} = -\epsilon_{\mu\nu}\, \partial_\mu (\boldsymbol{f} \cdot \partial_\nu \boldsymbol{m}). \tag{3.1}$$

We may now calculate the time derivative of the skyrmion number

$$\dot{\mathcal{N}} = \frac{1}{4\pi} \int \dot{n}\, d^2x = -\frac{1}{4\pi} \int \epsilon_{\mu\nu} \partial_\mu (\boldsymbol{f} \cdot \partial_\nu \boldsymbol{m})\, d^2x.$$

By using the divergence theorem the last integral gives a line integral at spatial infinity. Making the reasonable assumption that the integrated quantity falls rapidly enough at spatial infinity the integral is zero and thus $\dot{\mathcal{N}}$, i.e., the skyrmion number is a conserved quantity. We have thus given a direct proof of a result which was earlier given based on topological arguments.

It is very interesting that by extending the above calculations we can derive further conserved quantities which are written in terms of the topological density n. Let us consider the two moments of n

$$I_\mu = \int x_\mu\, n\, d^2x, \qquad \mu = 1, 2. \tag{3.2}$$

The time derivative is

$$\begin{aligned} \dot{I}_\mu &= \int x_\mu\, \dot{n}\, d^2x = -\epsilon_{\lambda\nu} \int x_\mu\, \partial_\lambda (\boldsymbol{f} \cdot \partial_\nu \boldsymbol{m})\, d^2x = \epsilon_{\mu\nu} \int \boldsymbol{f} \cdot \partial_\nu \boldsymbol{m}\, d^2x \\ &= \epsilon_{\mu\nu} \int [(\partial_k \partial_k \boldsymbol{m} - q m_3 \hat{\boldsymbol{e}}_3) \cdot \partial_\nu \boldsymbol{m}]\, d^2x. \end{aligned} \tag{3.3}$$

We have assumed that $\boldsymbol{f}$ contains only the exchange and anisotropy terms. We use that

$$m_3 \hat{\boldsymbol{e}}_3 \cdot \partial_\nu \boldsymbol{m} = m_3\, \partial_\nu m_3 = \frac{1}{2}\, \partial_\nu (m_3^2) = \delta_{\nu k} w_a$$

$$\begin{aligned} \partial_k \partial_k \boldsymbol{m} \cdot \partial_\nu \boldsymbol{m} &= \partial_k (\partial_k \boldsymbol{m} \cdot \partial_\nu \boldsymbol{m}) - \frac{1}{2} \partial_\nu (\partial_k \boldsymbol{m} \cdot \partial_k \boldsymbol{m}) = \partial_k \left[(\partial_k \boldsymbol{m} \cdot \partial_\nu \boldsymbol{m}) - \delta_{\nu k} \left(\frac{1}{2}\, \partial_\lambda \boldsymbol{m} \cdot \partial_\lambda \boldsymbol{m} \right) \right] \\ &= \partial_k \left[(\partial_k \boldsymbol{m} \cdot \partial_\nu \boldsymbol{m}) - \delta_{\nu k}\, w_e \right]. \end{aligned}$$

If we substitute the latter forms in the formula for the time derivative of I_μ in Eq. (3.3) we see that the integrand is in the form of a total derivative. The integral is equal to a line integral at spatial infinity by the divergence theorem. If the integrand $(\partial_k \boldsymbol{m} \cdot \partial_\nu \boldsymbol{m}) - \delta_{\nu k}\, w_e - \delta_{\nu k} w_a$ falls rapidly enough to zero at spatial infinity (i.e., if $\boldsymbol{m}$ goes to the ground state rapidly enough), then we find

$$\dot{I}_\mu = 0.$$

We have found that the moments of n in Eq. (3.2) are conserved quantities. It is possible to extend the present calculation and prove that I_μ are conserved for a general effective field $\boldsymbol{f}$, in particular, they are conserved in the presence of a magnetostatic field $\boldsymbol{h}_{\rm m}$ [**5**].

For the interpretation of the results of this section we should first note that the topological density takes significant values at the area of a vortex core or a bubble. That implies directly that the I_μ, which give the mean position of the topological density, are a measure of the position of the position of a vortex or bubble or of another topological soliton. More precisely, we define the so-called *guiding center* coordinates

$$R_x = \frac{I_x}{4\pi\mathcal{N}} = \frac{\int x\, n\, d^2x}{\int n\, d^2x}, \qquad R_y = \frac{I_y}{4\pi\mathcal{N}} = \frac{\int y\, n\, d^2x}{\int n\, d^2x}, \tag{3.4}$$

which can be taken to define the position of a bubble (or vortex), in the case $\mathcal{N} \neq 0$. The results of this section assert that the position of the bubble as defined in Eq. (3.4) is conserved during motion. That is, a bubble (or a vortex) is spontaneously pinned in a magnetic film and cannot be found in free translational motion.

EXERCISE 3.1. *Include an external magnetic field* $\boldsymbol{h}_{\rm ext} = (0, 0, h_{\rm ext})$ *in the effective field* $\boldsymbol{f}$ *of the LL equation and extend the calculation in Eq.* (3.3) *to find that, for a magnetic bubble,*

$$\dot{I}_\mu = -\int (\epsilon_{\mu\nu}\partial_\nu h_{\rm ext})(m_3 - 1)\, d^2x. \quad \square$$

4. Vortex-antivortex pairs

Let us consider the case of easy-plane anisotropy where the relevant topological excitations are vortices. An interesting object can be created if we assume that we have in a film a vortex and an antivortex in proximity to each other. We call this a vortex-ativortex (VA) pair. Its most important feature is that the magnetization is approaching a constant value at spatial infinity, e.g., $\boldsymbol{m}(\rho \to \infty) \to (1, 0, 0)$. This is because the in-plane phases of the vortex and the antivortex configurations cancel. Therefore we expect that, unlike a single vortex or antivortex, a VA pair has finite energy. A numerical simulation of magnetic vortices is shown in Fig. 1.

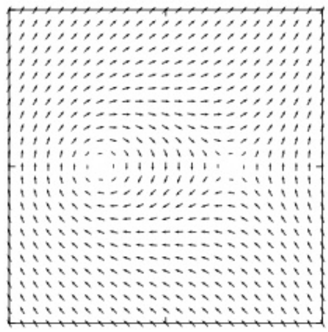

FIGURE 1. Numerical simulation of a magnetic vortex pair. Vectors give the in-plane component of the magnetization vector (m_1, m_2).

EXAMPLE 4.1. *Write an ansatz for a VA pair where the vortex and the antivortex have opposite polarities. Explain why this VA pair cannot propagate freely as a solitary wave.*

EXAMPLE 4.2. *Write an ansatz for a VA pair where the vortex and the antivortex have the same polarity* λ. *Explain why this VA pair may be a propagating solitary wave.*

CHAPTER 6

Numerical methods

1. Relaxation algorithms

Let us suppose a hamiltonian system with a pair of conjugate variables π, ϕ. If the energy functional is $E = E(\pi, \phi)$ then Hamilton's equations read

$$\dot{\pi} = \frac{\partial E}{\partial \phi}, \qquad \dot{\phi} = -\frac{\partial E}{\partial \pi}. \tag{1.1}$$

We can easily verify that the energy is conserved

$$\frac{dE}{dt} = \frac{\partial E}{\partial \pi}\frac{d\pi}{dt} + \frac{\partial E}{\partial \phi}\frac{d\phi}{dt} = \frac{\partial E}{\partial \pi}\frac{\partial E}{\partial \phi} - \frac{\partial E}{\partial \phi}\frac{\partial E}{\partial \pi} = 0.$$

It is sometimes useful to have an algorithm which would be able to find the minimum of the energy, that would be a *static* solution of Hamilton's equations. This can be achieved by using the following equations

$$\dot{\pi} = -\frac{dE}{d\pi}, \qquad \dot{\phi} = -\frac{dE}{d\phi}. \tag{1.2}$$

We can easily see that, under the latter equations of motion, the energy is decreasing for all $t > 0$:

$$\frac{dE}{dt} = \frac{\partial E}{\partial \pi}\frac{d\pi}{dt} + \frac{\partial E}{\partial \phi}\frac{d\phi}{dt} = -\left[\left(\frac{\partial E}{\partial \pi}\right)^2 + \left(\frac{\partial E}{\partial \phi}\right)^2\right] < 0.$$

Therefore, the energy is continuously decreasing until the conditions $\partial E/\partial\phi = 0 = \partial E/\partial\pi$ are satisfied. That is, the algorithm converges to a static solution of Hamilton's equations (1.1).

2. Stretched coordinates

It is sometimes useful to solve differential equations in non-uniform grids. Such a case arises, for example, when we need to solve an equation for $-\infty < x < \infty$. In order to formulate a numerical method, we may use a stretched coordinate ξ where

$$x = f(\xi).$$

As an example let us use

$$x = a\tan(\xi), \qquad -\frac{\pi}{2} \leq \xi \leq \frac{\pi}{2}.$$

If we use for ξ an equally spaced lattice with N points ($N-1$ intervals) then the spacing is

$$\Delta\xi = \frac{\pi}{N-1},$$

so that ξ takes the values

$$\xi_k = k\Delta\xi, \qquad k = -\frac{N-1}{2}, \ldots, -2, -1, 0, 1, 2, \ldots, \frac{N-1}{2},$$

where the first and last values for k give $\xi = \pm\pi/2 \Rightarrow x = \pm\infty$.

The lattice spacing in the cartesian (x) coordinate is non-uniform. For example, the lattice spacing at $x = 0$ is

$$\Delta x(x=0) = a\tan\left(\frac{\pi}{N-1}\right) \approx a\frac{\pi}{N-1},$$

while the lattice spacing at $\xi = \pi/3$ (that is, $x = a\tan(\pi/3)$) is

$$\Delta x(x = a\tan(\pi/3)) \approx \left.\frac{dx}{d\xi}\right|_{\frac{\pi}{3}} \Delta\xi = \frac{a}{\cos^2\left(\frac{\pi}{3}\right)}\frac{\pi}{N-1} \approx \frac{\Delta x(x=0)}{\cos^2\left(\frac{\pi}{3}\right)} = 4\Delta x(x=0).$$

The derivatives in stretched coordinates are calculated at position $x_i = f(\xi_i)$ as

$$\frac{\partial u}{\partial x}(x_i) = \frac{d\xi}{dx}\frac{\partial u}{\partial \xi} \approx \left.\frac{d\xi}{dx}\right|_i \frac{u_{i+1}-u_{i-1}}{2\Delta\xi}$$

and

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2}(x_i) &= \frac{\partial}{\partial x}\left[\frac{d\xi}{dx}\frac{\partial u}{\partial \xi}\right]\frac{d\xi}{dx}\frac{\partial}{\partial \xi}\left[\frac{d\xi}{dx}\frac{\partial u}{\partial \xi}\right] \approx \frac{1}{\Delta\xi}\left.\frac{d\xi}{dx}\right|_i \left[\left.\frac{\partial \xi}{\partial x}\right|_{i+\frac{1}{2}}\left.\frac{\partial f}{\partial \xi}\right|_{i+\frac{1}{2}} - \left.\frac{\partial \xi}{\partial x}\right|_{i-\frac{1}{2}}\left.\frac{\partial f}{\partial \xi}\right|_{i-\frac{1}{2}}\right] \\ &\approx \frac{1}{(\Delta\xi)^2}\left.\frac{d\xi}{dx}\right|_i \left[\left.\frac{\partial \xi}{\partial x}\right|_{i+\frac{1}{2}}(u_{i+1}-u_i) - \left.\frac{\partial \xi}{\partial x}\right|_{i-\frac{1}{2}}(u_i - u_{i-1})\right],\end{aligned}$$

where

$$\frac{dx}{d\xi} = \frac{a}{\cos^2(\xi)}.$$

CHAPTER 7

Projects

Notes:

- Project work must be structured, i.e., (a) first pose the problem, (b) write an introduction, (c) explain the relevant theory, (d) finally work on a particular problem and give systematic results (e.g., explore for different parameter values).
- Give particular examples using realistic (experimental) parameters (e.g., find a particular quality factor, or wall velocity, or vortex size, etc)
-

Consider one of the following projects (probably adjusted somewhat) or even suggest your own project.

1. Angles Θ, Φ

- Starting from the Landau-Lifshitz-Gilbert equation derive (giving details) the equations for the angle variables Θ, Φ (or Π, Φ).
- Derive the energy (exchange and anisotropy) in terms of Θ, Φ.
- Write the form of the equations for an axially symmetric vortex.

Hint: Use

$$\begin{aligned}
\frac{\delta W}{\delta \Pi} &= -\frac{1}{\sin\Theta}\frac{\delta W}{\delta \Theta} = -\frac{1}{\sin\Theta}\left[\frac{\delta W}{\delta m_1}\frac{\partial m_1}{\partial \Theta} + \frac{\delta W}{\delta m_2}\frac{\partial m_2}{\partial \Theta} + \frac{\delta W}{\delta m_3}\frac{\partial m_3}{\partial \Theta}\right] \\
\frac{\delta W}{\delta \Phi} &= \frac{\delta W}{\delta m_1}\frac{\partial m_1}{\partial \Phi} + \frac{\delta W}{\delta m_2}\frac{\partial m_2}{\partial \Phi} + \frac{\delta W}{\delta m_3}\frac{\partial m_3}{\partial \Phi}
\end{aligned}$$

2. Stereographic variable Ω

Do some of the following (or all of them, if you have enough time)

- Starting from the Landau-Lifshitz equation derive (giving details) the equation of motion Eq. (4.3) for the stereographic variable Ω.
- Derive the exchange energy in terms of Ω, Eq. (4.5).
- Write explicitly the equation for Ω.
- Verify that $\Omega = x + iy$ is a solution of the equation of motion (when only exchange is present).
- Explain what magnetic configuration the above simple solution represents.

Hint: The full equation (exchange, anistoropy, external field) reads

$$\begin{aligned}
(i+\alpha)\,\frac{\partial \Omega}{\partial t} &= \partial_\mu \partial_\mu \Omega - \frac{2\overline{\Omega}}{1+\Omega\overline{\Omega}}\,\partial_\mu \Omega\,\partial_\mu \Omega + q\,\frac{1-\Omega\overline{\Omega}}{1+\Omega\overline{\Omega}}\,\Omega \\
&\quad + \frac{h_1}{2}\,(1-\Omega^2) + i\frac{h_2}{2}\,(1+\Omega^2) - h_3\,\Omega.
\end{aligned}$$

3. Propagating domain wall

- Repeat the derivation of the solution for a propagating domain wall (giving some of the calculational details missing in the notes).
- Study the domain wall velocity for all possible Φ_0 and all possible quality factors q. Give corresponding plots. Explain qualitative differences between the cases.

4. Propagating domain wall using Ω

- Derive the solution for a propagating domain wall using the stereographic variable Ω.
- Study the domain wall velocity for all possible Φ_0 and all possible quality factors q. Give corresponding plots. Explain qualitative differences between the cases.

Hint: use the equation (but, explain what each term in the equation stands for)

$$(i+\alpha)\,\dot{\Omega} = \Omega_{xx} - \frac{2\overline{\Omega}}{1+\Omega\overline{\Omega}}\,(\Omega_x)^2 - q\,\frac{1-\Omega\overline{\Omega}}{1+\Omega\overline{\Omega}}\,\Omega - \frac{1}{2}\,\frac{\Omega+\overline{\Omega}}{1+\Omega\overline{\Omega}}\,(1-\Omega^2) - h\Omega,$$

and seek solutions of the form

$$\Omega = \Omega_0\, e^{i\Phi_0} = e^{\epsilon(x-vt)}\, e^{i\Phi_0}, \qquad \Omega_0 \equiv e^{\epsilon(x-vt)}, \quad \Phi_0 : \text{const.},$$

5. Domain wall: numerical simulation

- Simulate numerically a static and a propagating domain wall (use any computer language).
- Repeat the simulation for at least two cases (different q or velocity) and explain the differences.

6. Vortex profile

- Write the equation for a static axially symmetric magnetic vortex in cylindrical coordinates.
- Solve numerically the equations and find the static vortex profile.

7. Vortex in a ring particle

Suppose a ring-shaped magnetic element with easy-plane anisotropy.

- Write the form of a vortex configuration in the particle.
- Calculate exchange, anisotropy and magnetostatic energy.
- See whether this is a static solution of the equations of motion.
- Find experimental images of such ring particles (on the internet).

Bibliography

[1] S. Blundell, *"Magnetism in Condensed Matter"*, (OUP, Oxford, 2001)
[2] R. K. Wangsness, "Electromagnetic Fields", (John Wiley, 1979)
[3] J. D. Logan, *"Applied Mathematics"* (Wiley-Interscience, 1996)
[4] T. H. O' Dell, *"Ferromagnetodynamics"*, (John Wiley, New York, 1981).
[5] N. Papanicolaou, T. Tomaras, *"Dynamics of magnetic vortices"*, Nucl. Phys. B **360**, 425-462 (1991).
[6] S. Komineas, N. Papanicolaou, *"topology and dynamics in ferromagnetic media"*, Physica D **81**, 81-107, (1996).
[7] E. M. Lifshitz and L. P. Pitaevskii, *"Statistical Physics, Part 2"*, (Pergamon, 3rd edition, 1980).
[8] P. G. Saffman, *"Vortex Dynamics"*, (Cambridge University Press, 1992), (chapter 7).
[9] H. Aref, *Point vortex dynamics: A classical mathematics playground*, J. Math. Phys. **48**, 065401 (2007).
[10] A. P. Malozemoff, J. C Slonczewski, *Magnetic domain walls in bubble materials* (Academic Press, 1979)
[11] M. Nakahara, *Geometry, Topology and Physics* (IOP Publishing, 1990)
[12] R. Rajaraman, *Solitons and Instantons* (North Hollad, 1982)